MATHEMATICAL APPENDIX

- **A. The Model**

- The representative consumer solves

$$\max_{\{C(t),\, u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t) dt,$$

subject to

$$\text{(A)}\quad \dot{K}(t) = \left[1 - \alpha\tau_K - (1-\alpha)\tau_L + \frac{(1-\alpha)su(t)}{1-u(t)}\right] Y(t) - (1+\tau_C)C(t) - \delta_K K(t),$$

$$\text{(B)}\quad \dot{h}(t) = B(t)\left[u(t)h(t)\right]^{\phi} \overline{h}(t)^{\theta} - (\delta_h + g_L)h(t),$$

for all $t > 0$, with $\dot{A}(t) = g_A A(t)$, $\dot{B}(t) = g_B B(t)$,and $\dot{L}(t) = g_L L(t)$, given $(K(0), h(0), A(0), B(0), L(0))$ and $\{\overline{h}(t)\}$ for $t \geq 0$.

- Notice that the human capital production function is $B(t)\left[u(t)h(t)\right]^{\phi} \overline{h}(t)^{\theta}$. It was To $B\left[A(t)u(t)h(t)\right]^{\phi} \overline{h}(t)^{\theta}$ in the previous draft, which is still a special case that is considered in the current draft.

- **B. FOCs**

- Since $A(t) = e^{g_A t}A(0)$ and $B(t) = e^{g_B t}B(0)$, I have

$$\frac{B(t)}{A(t)} = e^{(g_B - g_A)t}\frac{B(0)}{A(0)}.$$

$$\text{So } B(t) = e^{(g_B - g_A)t}\frac{B(0)}{A(0)}A(t)$$

- Write (B) as

$$\dot{h}(t) = e^{(g_B - g_A)t}\frac{B(0)}{A(0)}A(t)\left[u(t)h(t)\right]^{\phi} \overline{h}(t)^{\theta} - (\delta_h + g_L)h(t).$$

- First, find how the problem can be transformed into an intensive form.

- Define:

$$c = \frac{C}{A^{\beta}L},$$

$$k = \frac{K}{A^{\beta}L},$$

$$\tilde{h} = \frac{h}{A^{\beta-1}}.$$

- So

$$\text{(A)}\quad \frac{\dot{K}}{A^\beta L} = \left[\underbrace{1-\alpha\tau_K-(1-\alpha)\tau_L}_{\equiv d} + \frac{(1-\alpha)su}{1-u}\right]\frac{Y}{A^\beta L} - (1+\tau_C)\frac{C}{A^\beta L} - \delta_K \frac{K}{A^\beta L}$$
$$= \left[d + \frac{(1-\alpha)su}{1-u}\right]\left(\frac{K}{A^\beta L}\right)^\alpha \left(\frac{(1-u)h}{A^{\beta-1}}\right)^{1-\alpha} - (1+\tau_C)c - \delta_K k$$
$$= \left[d + \frac{(1-\alpha)su}{1-u}\right] k^\alpha (1-u)^{1-\alpha}\tilde{h}^{1-\alpha} - (1+\tau_C)c - \delta_K k$$

- Also,

$$\text{(B)}\quad \frac{\dot{h}}{A^{\beta-1}} = \frac{e^{(g_B-g_A)t}\frac{B(0)}{A(0)}A\,[uh]^\phi\,\overline{h}^\theta}{A^{\beta-1}} - (\delta_h+g_L)\frac{h}{A^{\beta-1}}$$
$$= \frac{\frac{B(0)}{A(0)}\left[uA^{\beta-1}\tilde{h}\right]^\phi \left[A^{\beta-1}\tilde{\overline{h}}\right]^\theta}{e^{-(g_B-g_A)t}A^{\beta-2}} - (\delta_h+g_L)\tilde{h}$$
$$= \frac{\frac{B(0)}{A(0)}\left[u\tilde{h}\right]^\phi \left[\tilde{\overline{h}}\right]^\theta}{e^{-(g_B-g_A)t}A^{\beta-2-(\beta-1)(\phi+\theta)}} - (\delta_h+g_L)\tilde{h}$$

- So the intensive form works if $e^{-(g_B-g_A)t}A^{\beta-2-(\beta-1)(\phi+\theta)}$ is constant. This holds if

$$-(g_B-g_A) + g_A(\beta-2-(\beta-1)(\phi+\theta)) = 0.$$
$$\text{So } g_A(\beta-2-(\beta-1)(\phi+\theta)) = g_B - g_A.$$
$$\text{So } \beta-2-(\beta-1)(\phi+\theta) = \frac{g_B}{g_A} - 1.$$
$$\text{So } \beta(1-\phi-\theta) + (\phi+\theta) = \frac{g_B}{g_A} + 1.$$
$$\text{So } \beta(1-\phi-\theta) = \frac{g_B}{g_A} + 1 - \phi - \theta.$$
$$\text{So } \beta = \frac{g_B}{g_A}\frac{1}{1-\phi-\theta} + 1.$$

- In the previous draft, the human capital production function was $B\left[A(t)u(t)h(t)\right]^\phi \overline{h}(t)^\theta$. This is a special case of the current version's function, $B(t)\left[u(t)h(t)\right]^\phi \overline{h}(t)^\theta$, in which $B(t) = BA(t)^\phi$, implying $g_B = \phi g_A$. Plugging this into the above equation, I have $\beta = \frac{\phi g_A}{g_A}\frac{1}{1-\phi-\theta} + 1 = \frac{\phi}{1-\phi-\theta} + 1 = \frac{1-\theta}{1-\phi-\theta}$. This is the same as in the previous draft.

- In other words,

$$c = \frac{C}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L},$$
$$k = \frac{K}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L},$$
$$\tilde{h} = \frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}}.$$

- This implies

$$K = kA^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L,$$
$$h = \tilde{h}A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}.$$

- So

$$\dot{K} = \dot{k}A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)kA^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}\dot{A}L + kA^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}\dot{L},$$
$$\dot{h} = \dot{\tilde{h}}A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}} + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}\right)\tilde{h}A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}-1}\dot{A}.$$

- Therefore,

$$\text{(A)}\ \frac{\dot{K}}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L} = \dot{k} + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)g_A k + g_L k$$
$$= \left[d + \frac{(1-\alpha)su}{1-u}\right]k^{\alpha}(1-u)^{1-\alpha}\tilde{h}^{1-\alpha} - (1+\tau_C)c - \delta_K k$$

So

$$\text{(A)}\ \dot{k} = \left[d + \frac{(1-\alpha)su}{1-u}\right]k^{\alpha}(1-u)^{1-\alpha}\tilde{h}^{1-\alpha}$$
$$-(1+\tau_C)c - \underbrace{\left[\delta_K + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)g_A + g_L\right]}_{\equiv e}k$$

- Also,

$$\text{(B)}\ \frac{\dot{h}}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}} = \dot{\tilde{h}} + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}\right) g_A \tilde{h} = \dot{\tilde{h}} + \frac{g_B}{1-\phi-\theta}\tilde{h}$$

$$= \frac{\frac{B(0)}{A(0)}\left[u\tilde{h}\right]^{\phi}\widetilde{\overline{h}}^{\theta}}{e^{-(g_B-g_A)t}A^{\beta-2-(\beta-1)(\phi+\theta)}} - (\delta_h + g_L)\tilde{h}$$

$$= \frac{\frac{B(0)}{A(0)}\left[u\tilde{h}\right]^{\phi}\widetilde{\overline{h}}^{\theta}}{A(0)^{\beta-2-(\beta-1)(\phi+\theta)}} - (\delta_h + g_L)\tilde{h}$$

$$= \frac{B(0)}{A(0)^{\beta-1-(\beta-1)(\phi+\theta)}}\left[u\tilde{h}\right]^{\phi}\widetilde{\overline{h}}^{\theta} - (\delta_h + g_L)\tilde{h}$$

$$= \frac{B(0)}{A(0)^{(\beta-1)(1-\phi-\theta)}}\left[u\tilde{h}\right]^{\phi}\widetilde{\overline{h}}^{\theta} - (\delta_h + g_L)\tilde{h}$$

$$= \frac{B(0)}{A(0)^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}(1-\phi-\theta)}}\left[u\tilde{h}\right]^{\phi}\widetilde{\overline{h}}^{\theta} - (\delta_h + g_L)\tilde{h}$$

$$= \underbrace{\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}}_{\equiv \bar{B}}\left[u\tilde{h}\right]^{\phi}\widetilde{\overline{h}}^{\theta} - (\delta_h + g_L)\tilde{h}$$

- So

$$\text{(B)}\ \dot{\tilde{h}} = \bar{B}u^{\phi}\tilde{h}^{\phi}\widetilde{\overline{h}}^{\theta} - \underbrace{\left(\delta_h + g_L + \frac{g_B}{1-\phi-\theta}\right)}_{\equiv f}\tilde{h}$$

- Objective:

$$\max_{\{C_t,u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{[C/L]^{1-\sigma}}{1-\sigma}L dt$$

$$= \max_{\{c_t,u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{[cA^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}]^{1-\sigma}}{1-\sigma}L_t dt$$

$$= \max_{\{c_t,u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma}A^{\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma)} L dt$$

$$= \max_{\{c_t,u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma}\underbrace{A(0)^{\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma)}}_{\text{const}} e^{g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma)t}\underbrace{L(0)}_{\text{const}}e^{g_L t}dt$$

$$\Longrightarrow \max_{\{c_t,u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\left(\rho - g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma) - g_L\right)t}\frac{c^{1-\sigma}}{1-\sigma}dt$$

$$= \max_{\{c_t,u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t}\frac{c^{1-\sigma}}{1-\sigma}dt$$

where

$$\eta \equiv \rho - g_A \left(\frac{g_B}{g_A} \frac{1}{1-\phi-\theta} + 1 \right) (1-\sigma) - g_L.$$

$\eta > 0$ is satisfied with calibrated values.

- Set up Hamiltonian with this objective and (A) and (B):

$$\begin{aligned} H = & \frac{c^{1-\sigma}}{1-\sigma} \\ & + \lambda \left(\left[d + \frac{(1-\alpha)su}{1-u} \right] k^\alpha (1-u)^{1-\alpha} \tilde{h}^{1-\alpha} - (1+\tau_C)c - ek \right) \\ & + \mu \left[\bar{B} u^\phi \tilde{h}^\phi \widetilde{\overline{h}}^\theta - f \tilde{h} \right] \end{aligned}$$

- FOCs are

$$\begin{aligned} (1)\ & H_c = 0, \\ (2)\ & H_u = 0, \\ (3)\ & H_k = \eta\lambda - \dot{\lambda}, \\ (4)\ & H_{\tilde{h}} = \eta\mu - \dot{\mu}. \end{aligned}$$

- So

$$(1)\ c^{-\sigma} - (1+\tau_C)\lambda = 0.$$

- And

$$\begin{aligned} (2)\ 0 = & \lambda \left(\frac{1}{1-u} + \frac{u}{(1-u)^2} \right) s(1-\alpha) k^\alpha [(1-u)\tilde{h}]^{1-\alpha} \\ & - \lambda \left(d + \frac{u}{1-u} s(1-\alpha) \right) k^\alpha (1-\alpha)(1-u)^{-\alpha} \tilde{h}^{1-\alpha} \\ & + \mu \bar{B} \phi u^{\phi-1} \tilde{h}^\phi \widetilde{\overline{h}}^\theta \end{aligned}$$

Rearranging $\frac{1}{1-u} + \frac{u}{(1-u)^2}$,

$$\begin{aligned} (2)\ 0 = & \lambda \frac{1}{(1-u)^2} s(1-\alpha) k^\alpha [(1-u)\tilde{h}]^{1-\alpha} \\ & - \lambda \left(d + \frac{u}{1-u} s(1-\alpha) \right) k^\alpha (1-\alpha)(1-u)^{-\alpha} \tilde{h}^{1-\alpha} \\ & + \mu \bar{B} \phi u^{\phi-1} \tilde{h}^\phi \widetilde{\overline{h}}^\theta \end{aligned}$$

Now I incorporate the first two terms:

$$(2)\ 0 = \lambda \frac{1}{1-u} s(1-\alpha) k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha} \\ - \lambda \left(d + \frac{u}{1-u} s(1-\alpha) \right) k^{\alpha}(1-\alpha)(1-u)^{-\alpha}\tilde{h}^{1-\alpha} \\ + \mu \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi}\widetilde{\overline{h}}^{\theta}$$

or

$$(2)\ 0 = -\lambda \left[d + \frac{u}{1-u} s(1-\alpha) - \frac{1}{1-u} s \right] (1-\alpha) k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha} \\ + \mu \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi}\widetilde{\overline{h}}^{\theta}$$

or

$$(2)\ 0 = -\lambda \left[d + \frac{su - \alpha su - s}{1-u} \right] (1-\alpha) k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha} \\ + \mu \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi}\widetilde{\overline{h}}^{\theta}$$

or

$$(2)\ 0 = -\lambda \left[d - s - \frac{\alpha su}{1-u} \right] (1-\alpha) k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha} \\ + \mu \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi}\widetilde{\overline{h}}^{\theta}$$

- And

$$(3)\ \lambda \left(\left[d + \frac{u}{1-u} s(1-\alpha) \right] \alpha k^{\alpha-1}[(1-u)\tilde{h}]^{1-\alpha} - e \right) = \eta\lambda - \dot{\lambda}$$

- And

$$(4)\ \lambda \left[d + \frac{u}{1-u} s(1-\alpha) \right] k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha} + \mu \left[\bar{B}u^{\phi}\phi\tilde{h}^{\phi-1}\widetilde{\overline{h}}^{\theta} - f \right] = \eta\mu - \dot{\mu}$$

- To summarize,

$$(1)\ c^{-\sigma} - (1+\tau_C)\lambda = 0$$

$$(2)\ -\lambda \left[d - s - \frac{\alpha su}{1-u} \right] (1-\alpha) k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha} + \mu \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi}\widetilde{\overline{h}}^{\theta} = 0$$

$$(3)\ \lambda \left(\left[d + \frac{u}{1-u} s(1-\alpha) \right] \alpha k^{\alpha-1}[(1-u)\tilde{h}]^{1-\alpha} - e \right) = \eta\lambda - \dot{\lambda}$$

$$(4)\ \lambda \left[d + \frac{u}{1-u} s(1-\alpha) \right] k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha} + \mu \left[\bar{B}u^{\phi}\phi\tilde{h}^{\phi-1}\widetilde{\overline{h}}^{\theta} - f \right] = \eta\mu - \dot{\mu}$$

$$(A)\ \dot{k} = \left[d + \frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}[(1-u)\tilde{h}]^{1-\alpha} - (1+\tau_C)c - ek$$

$$(B)\ \dot{\tilde{h}} = \bar{B}u^{\phi}\tilde{h}^{\phi}\tilde{\bar{h}}^{\theta} - f\tilde{h}$$

- **C. BGP**
- BGP: Assume all variables grow at constant rates (or stay at constant levels).
- <Step 0> (1) implies

$$-\sigma\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}.$$

$$\text{So (1')}\ g_c = -\frac{1}{\sigma}g_{\lambda}.$$

- <Step 1> (B) implies that after $\tilde{h} = \tilde{\bar{h}}$ is imposed,

$$(B')\ \frac{\dot{\tilde{h}}}{\tilde{h}} = \bar{B}u^{\phi}\tilde{h}^{\phi+\theta-1} - f.$$

Here, I further assume that u is constant. If u has a strictly positive growth rate, then it will eventually violate $u < 1$. If u has a strictly negative growth rate, then it will converge to 0, which is not consistent with data. Assuming u is constant, the above equation implies that I either have $\phi + \theta = 1$ or a constant $\tilde{h}$. Recall that I already assumed $\phi + \theta < 1$, implying that is $\tilde{h}$ constant.

- To see that $\tilde{h}$ is constant in another way, consider (B) after imposing $h = \bar{h}$:

$$(B)\ \ \dot{h} = B\left[uh\right]^{\phi}h^{\theta} - (\delta_h + g_L)h.$$

$$\text{So } g_h = Bu^{\phi}h^{\phi+\theta-1} - (\delta_h + g_L).$$

- This implies

$$g_B + (\phi+\theta-1)g_h = 0$$

on BGP. This implies $\tilde{h} = \frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}} = \frac{h}{(\text{const})e^{g_A\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}t}} = \frac{h}{(\text{const})e^{\frac{g_B}{1-\phi-\theta}t}} = \frac{h}{(\text{const})e^{g_h t}}$ which is constant.

- So $\boxed{g_{\tilde{h}} = 0}$. Hence,

$$(B')\ g_{\tilde{h}} = 0 = \bar{B}u^{\phi}\tilde{h}^{\phi+\theta-1} - f.$$

- <Step 2> (3):

$$(3)\ \lambda\left(\left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha k^{\alpha-1}[(1-u)\tilde{h}]^{1-\alpha}-e\right)=\eta\lambda-\dot{\lambda}$$

$$(3')\ \left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha k^{\alpha-1}[(1-u)\tilde{h}]^{1-\alpha}-e=\eta-\frac{\dot{\lambda}}{\lambda}$$

So $k/\tilde{h}$ should be constant, implying that $\boxed{g_k = g_{\tilde{h}} = 0}$.

- <Step 3> (A):

$$(A)\ \dot{k}=0=\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}[(1-u)\tilde{h}]^{1-\alpha}-(1+\tau_C)c-ek$$

Since everything else is constant, c should be constant, implying that $\boxed{g_c = g_k = g_{\tilde{h}} = 0}$.

- <Step 4> (2) implies that after $\tilde{h}=\tilde{\tilde{h}}$ is imposed,

$$(2)\ -\lambda\left[d-s-\frac{\alpha s u}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}+\mu\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi}\tilde{\tilde{h}}^{\theta}=0$$

$$(2')\ \lambda\left[d-s-\frac{\alpha s u}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}=\mu\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta}$$

Since everything else is constant, $\boxed{g_\lambda = g_\mu}$.

- But recall that from (1'), $g_c = -\frac{1}{\sigma}g_\lambda$. Finally, $\boxed{g_c = g_k = g_{\tilde{h}} = g_\lambda = g_\mu = 0}$.

- **D. Elimination of $\lambda(t)$ and $\mu(t)$**

- From (1), (1'):

$$(1)\ c^{-\sigma}=(1+\tau_C)\lambda, \quad (1')\ g_c=0.$$

This can be used to eliminate λ.

- From (2'):

$$(2')\ \lambda\left[d-s-\frac{\alpha s u}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}=\mu\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta}$$

$$(2'')\ \frac{\left[d-s-\frac{\alpha s u}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}}{\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta}}=\frac{\mu}{\lambda}$$

- From (3'): λ is eliminated as

$$(3')\quad \left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha k^{\alpha-1}[(1-u)\tilde{h}]^{1-\alpha}-e=\eta-\frac{\dot{\lambda}}{\lambda}$$

$$(3'')\quad \left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}\left(\frac{k}{\tilde{h}}\right)^{\alpha-1}-e=\eta$$

- (4):

$$(4)\ \lambda\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha}+\mu\left[\bar{B}u^{\phi}\phi\tilde{h}^{\phi-1}\overline{\tilde{h}}^{\theta}-f\right]=\eta\mu-\dot{\mu}$$

$$\text{So } \frac{\lambda}{\mu}\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha}+\bar{B}u^{\phi}\phi\tilde{h}^{\phi-1}\overline{\tilde{h}}^{\theta}-f=\eta$$

Using (2"),

$$\frac{\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha}}{\frac{\left[d-s-\frac{\alpha su}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}}{\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta}}}+\bar{B}u^{\phi}\phi\tilde{h}^{\theta+\phi-1}-f=\eta$$

$$\text{So } \frac{\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta}\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha}}{\left[d-s-\frac{\alpha su}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}}+\bar{B}u^{\phi}\phi\tilde{h}^{\theta+\phi-1}-f=\eta$$

$$\text{So } \frac{\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta-1}\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha}}{\left[d-s-\frac{\alpha su}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{-\alpha}}+\bar{B}u^{\phi}\phi\tilde{h}^{\theta+\phi-1}-f=\eta$$

$$\text{So } \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta-1}\left[\frac{\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}(1-u)^{1-\alpha}(1-\alpha)\tilde{h}^{-\alpha}}{\left[d-s-\frac{\alpha su}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{-\alpha}}+u\right]-f=\eta$$

The part in [] becomes

$$
\begin{aligned}
&\frac{\left[d+\frac{u}{1-u}s(1-\alpha)\right](1-u)}{d-s-\frac{\alpha su}{1-u}}+u \\
&= \frac{\left[d+\frac{u}{1-u}s(1-\alpha)\right]-u\left[d+\frac{u}{1-u}s(1-\alpha)\right]+u\left[d-s-\frac{\alpha su}{1-u}\right]}{d-s-\frac{\alpha su}{1-u}} \\
&= \frac{\left[d+\frac{u}{1-u}s(1-\alpha)\right]-u\left[\frac{u}{1-u}s(1-\alpha)\right]+u\left[-s-\frac{\alpha su}{1-u}\right]}{d-s-\frac{\alpha su}{1-u}} \\
&= \frac{d+\frac{u}{1-u}s(1-\alpha)-u\frac{u}{1-u}s(1-\alpha)-su-u\frac{\alpha su}{1-u}}{d-s-\frac{\alpha su}{1-u}} \\
&= \frac{d+\frac{su-\alpha su-su^2+\alpha su^2-su+su^2-\alpha su^2}{1-u}}{d-s-\frac{\alpha su}{1-u}} \\
&= \frac{d-\frac{\alpha su}{1-u}}{d-\frac{\alpha su}{1-u}-s}
\end{aligned}
$$

So

$$
(4')\ \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta-1}\frac{d-\frac{\alpha su}{1-u}}{d-\frac{\alpha su}{1-u}-s}-f=\eta
$$

- Take (A) and (B')

$$
\begin{aligned}
&(A)\ 0=\left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}[(1-u)\tilde{h}]^{1-\alpha}-(1+\tau_C)c-ek \\
&(B')\ 0=\bar{B}u^{\phi}\tilde{h}^{\phi+\theta-1}-f.
\end{aligned}
$$

- Situation: I had 6 eqs. I eliminated λ and μ. I now have 4 eqs:

$$
\begin{aligned}
(3'')\ \left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}\left(\frac{k}{\tilde{h}}\right)^{\alpha-1}-e=\eta \\
(4')\ \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta-1}\frac{d-\frac{\alpha su}{1-u}}{d-\frac{\alpha su}{1-u}-s}-f=\eta \\
(A)\ \left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}[(1-u)\tilde{h}]^{1-\alpha}-(1+\tau_C)c-ek=0 \\
(B')\ \bar{B}u^{\phi}\tilde{h}^{\phi+\theta-1}-f=0
\end{aligned}
$$

- Existence and uniqueness of solutions: Given all other constants, the representative consumer chooses u, k, $\tilde{h}$ and c. Now I see whether the solution exists and is unique. Plugging (B') into (4'),

$$
\phi\frac{f}{u}\frac{d-\frac{\alpha su}{1-u}}{d-\frac{\alpha su}{1-u}-s}-f=\eta.
$$

The first task is to show that this has a unique solution, u. Rearranging,

$$\frac{d(1-u)-\alpha su}{(d-s)(1-u)-\alpha su}=\frac{f+\eta}{f\phi}u.$$

The LHS is a continuous function. Since $d=1-\alpha\tau_K-(1-\alpha)\tau_L>0$ and $s\geq 0$, LHS is a function, decreasing from $d/(d-s)>1$ to 1, for $0<u<1$. To see this, make a first derivative wrt u:

$$\begin{aligned}
&\frac{-d-\alpha s}{(d-s)(1-u)-\alpha su}-\frac{[d(1-u)-\alpha su](-d+s-\alpha s)}{[(d-s)(1-u)-\alpha su]^2}\\
&=-\frac{[(d-s)(1-u)-\alpha su](-d-\alpha s)-[d(1-u)-\alpha su](-d+s-\alpha s)}{[(d-s)(1-u)-\alpha su]^2}\\
&=-\frac{[d-du-s+su-\alpha su](-d-\alpha s)-[d-du-\alpha su](-d+s-\alpha s)}{[(d-s)(1-u)-\alpha su]^2}\\
&=\frac{[d-du-s+su-\alpha su](d+\alpha s)+[d-du-\alpha su](-d+s-\alpha s)}{[(d-s)(1-u)-\alpha su]^2}\\
&=\frac{d^2-d^2u-ds+dsu-\alpha dsu+\alpha ds-\alpha dsu-\alpha s^2+\alpha s^2u-\alpha^2s^2u}{[(d-s)(1-u)-\alpha su]^2}\\
&+\frac{-d^2+d^2u+\alpha dsu+ds-dsu-\alpha s^2u-\alpha ds+\alpha dsu+\alpha^2s^2u}{[(d-s)(1-u)-\alpha su]^2}\\
&=\frac{-\alpha s^2}{[(d-s)(1-u)-\alpha su]^2}<0
\end{aligned}$$

- The RHS changes from 0 to $(f+\eta)/f\phi$. Therefore, u has a unique solution in $(0,1)$ if and only if

  $$(f+\eta)/f\phi>1.$$

  By definition, $f=\delta_h+g_L+\frac{\phi g_A}{1-\phi-\theta}>0$ and $\phi>0$. So this is equivalent to

  $$\text{(Condition 1) } f+\eta>f\phi.$$

- Once a solution $0<u<1$ is given, (B') gives a strictly positive solution for $\tilde{h}$:

  $$\tilde{h}=\left(\frac{f}{\bar{B}u^{\phi}}\right)^{\frac{1}{\phi+\theta-1}}>0.$$

- Then, (3") gives a solution for k through:

  $$k=\tilde{h}\left(\frac{e+\eta}{\left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}.$$

This solution is strictly positive if and only if

$$\text{(Condition 2) } e + \eta > 0$$

- Finally, (A) gives a solution for c through

$$\left[d + \frac{u}{1-u}s(1-\alpha)\right] k^{\alpha}[(1-u)\tilde{h}]^{1-\alpha} - ek = (1+\tau_C)c$$

- So the solution for c is strictly positive if and only if

$$\left[d + \frac{u}{1-u}s(1-\alpha)\right] [(1-u)\frac{\tilde{h}}{k}]^{1-\alpha} > e$$

$$\text{So } \left[d + \frac{u}{1-u}s(1-\alpha)\right] [(1-u)\left(\frac{e+\eta}{\left[d + \frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}}]^{1-\alpha} > e$$

$$\text{So } \left[d + \frac{u}{1-u}s(1-\alpha)\right] (1-u)^{1-\alpha}\left(\frac{e+\eta}{\left[d + \frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}}\right) > e$$

$$\text{So } \left(\frac{e+\eta}{\alpha}\right) > e$$

$$\text{(Condition 3) } e + \eta > \alpha e$$

Since $\alpha e > 0$, Condition 3 is more strict than Condition 2.

- **Proposition 1**: The solutions for the consumer's problem on the balanced growth path, $c > 0$, $k > 0$, $\tilde{h} > 0$ and $0 < u < 1$, exist and are unique if and only if

$$f(1-\phi) + \eta > 0.$$
$$e(1-\alpha) + \eta > 0.$$

- These two conditions imply

$$\left(\delta_h + g_L - \frac{g_B}{1-\phi-\theta}\right)(1-\phi) + \rho > g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta} + 1\right)(1-\sigma) + g_L.$$

$$\left[\delta_K + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta} + 1\right)g_A + g_L\right](1-\alpha) + \rho > g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta} + 1\right)(1-\sigma) + g_L.$$

- These conditions are satisfied under the benchmark calibration result.

- **E. Recovery of Original Notations**

- Recall:

$$\begin{aligned}
c &= \frac{C}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L}, \\
g_c &= g_C - \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)g_A - g_L \\
k &= \frac{K}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L}, \\
\tilde{h} &= \frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}}, \\
\eta &= \rho - g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma) - g_L, \\
d &= 1 - \alpha\tau_K - (1-\alpha)\tau_L, \\
e &= \delta_K + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)g_A + g_L, \\
f &= \delta_h + g_L + \frac{g_B}{1-\phi-\theta}, \\
\bar{B} &= \frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}.
\end{aligned}$$

- **Part 0/4**: Regarding c above

$$\begin{aligned}
g_C &= g_c + \left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)g_A + g_L \\
&= 0 + g_A + \frac{g_B}{1-\phi-\theta} + g_L
\end{aligned}$$

- But in Step 1 of "C: BGP", I showed $g_B + (\phi+\theta-1)g_h = 0$. So

$$g_C = g_A + g_h + g_L$$

- Similarly,

$$g_K = g_A + g_h + g_L$$

- So from the production function $Y = K^\alpha[A(1-u)hL]^{1-\alpha}$,

$$\begin{aligned}
g_Y &= \alpha g_K + (1-\alpha)(g_A + g_h + g_L) \\
&= g_A + g_h + g_L
\end{aligned}$$

- **Part 1/4 (Regarding (3”))**: Let’s start with (3”):

$$(3\text{”})\quad \left[d+\frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}\left(\frac{k}{\tilde{h}}\right)^{\alpha-1}-e=\eta$$

- So

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}\left(\frac{\frac{K}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L}}{\frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}}}\right)^{\alpha-1}$$
$$=\delta_K+\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)g_A+g_L+\rho-g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma)-g_L$$

- So

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{u}{1-u}s(1-\alpha)\right]\alpha(1-u)^{1-\alpha}\left(\frac{K}{AhL}\right)^{\alpha-1}$$
$$=\delta_K+\rho+g_A\sigma\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)$$

- So

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]\frac{\alpha}{K/Y}-\delta_K$$
$$=\rho+g_A\sigma\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)$$

- In Part 0/4 above, I showed $g_C-g_L=g_A+g_h$. In Step 1 of "C: BGP", I showed $g_B+(\phi+\theta-1)g_h=0$. Hence, $g_C-g_L=g_A+\frac{g_B}{1-\phi-\theta}$. Hence,

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]\frac{\alpha}{K/Y}-\delta_K=\rho+\sigma\left(g_C-g_L\right)$$

- This is an Euler eq!

- Since K/Y is constant (which is clear from Part 0/4), I will treat (K/Y) as constant:

$$(3\text{”’})\quad \rho+\sigma\left(g_C-g_L\right)=\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]r-\delta_K$$
$$=\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]\frac{\alpha}{K/Y}-\delta_K$$

- The second eq. is the definition of r. (Simply, $r =$MPK before tax.)
- **Part 2/4 (Regarding (4'))**: Now (4'):

$$\text{(4')}\ \bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta-1}\frac{d-\frac{\alpha su}{1-u}}{d-\frac{\alpha su}{1-u}-s}-f=\eta$$

- So

$$\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}\phi u^{\phi-1}\left(\frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}}\right)^{\phi+\theta-1}\frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}$$
$$=\delta_h+g_L+\frac{g_B}{1-\phi-\theta}+\rho-g_A\left(\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1\right)(1-\sigma)-g_L$$

- So

$$\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}\phi u^{\phi-1}h^{\phi+\theta-1}A^{\frac{g_B}{g_A}}\frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}$$
$$=\delta_h+\frac{g_B\sigma}{1-\phi-\theta}+\rho-g_A(1-\sigma)$$

- So

$$\frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}\phi u^{\phi-1}h^{\phi+\theta-1}A^{\frac{g_B}{g_A}}-\delta_h+g_A$$
$$=\frac{g_B\sigma}{1-\phi-\theta}+\rho+g_A\sigma$$

- Again, $g_C-g_L=g_A+\frac{g_B}{1-\phi-\theta}$ (as I used in Part 1/4), so

$$\frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}\phi u^{\phi-1}h^{\phi+\theta-1}A^{\frac{g_B}{g_A}}-\delta_h+g_A$$
$$=\rho+\sigma(g_C-g_L)$$

- Recall that in "B: FOCs", I assumed $B(t)=e^{(g_B-g_A)t}\frac{B(0)}{A(0)}A(t)$. Hence,

$$\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}A^{\frac{g_B}{g_A}}=\frac{A(0)B}{e^{(g_B-g_A)t}A(0)^{\frac{g_B}{g_A}}A}A^{\frac{g_B}{g_A}}=\frac{B}{e^{(g_B-g_A)t}}\frac{A^{\frac{g_B}{g_A}-1}}{A(0)^{\frac{g_B}{g_A}-1}}$$
$$=\frac{B}{e^{(g_B-g_A)t}}e^{g_A\left(\frac{g_B}{g_A}-1\right)t}=B$$

implying that

$$\frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}\underbrace{B\phi u^{\phi-1}h^{\phi+\theta-1}}_{\text{(a)}}-\delta_h+g_A$$
$$=\rho+\sigma(g_C-g_L)$$

- What is (a)?: Recall
$$\dot{h}=B\,(uh)^{\phi}\,\overline{h}^{\theta}-(\delta_h+g_L)h.$$
The human capital production function is $B\,(uh)^{\phi}\,\overline{h}^{\theta}$. So MPH is
$$B\phi u^{\phi-1}h^{\phi-1+\theta}$$
after $\overline{h}=h$ is imposed. This is (a), i.e., MPH in human capital production.

- One may ask whether this MPH is based on per-capita human capital stock, not aggregate human capital stock, so it is not comparable to the MPK (interest rate). It turns out that MPH based on aggregate human capital stock is still the same. To see this, I multiply the H accumulation by L:
$$\dot{H}=B\,(uhL)^{\phi}\,(\overline{h}L)^{\theta}L^{1-\phi-\theta}-\delta_h H.$$
The MPH is therefore
$$B\phi\,(uhL)^{\phi-1}\,(\overline{h}L)^{\theta}L^{1-\phi-\theta}=B\phi u^{\phi-1}h^{\phi+\theta-1}$$
which is the same as before.

- Write it with (3"'):
$$\text{(3"')}\ \rho+\sigma\,(g_C-g_L)=\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]r-\delta_K$$
$$=\left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]\frac{\alpha}{K/Y}-\delta_K$$
$$=\frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}B\phi u^{\phi-1}h^{\phi+\theta-1}+g_A-\delta_h$$

- So
$$\begin{aligned}\text{(a function of consumption growth)}&=\text{(net interest rate)}-\delta_K\\&=\text{(net MPK in Y production)}-\delta_K\\&=\text{(net MPH in H production)}+g_A-\delta_h\end{aligned}$$

- ("Net" means after-tax-and-subsidy.)
- I can also manipulate the FOCs to relate (MPH in Y production) to here.
- From (2"):

$$(2") \quad \frac{\left[d - s - \frac{\alpha s u}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{1-\alpha}}{\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta}} = \frac{\mu}{\lambda}$$

$$\text{So} \quad \frac{\left[d - s - \frac{\alpha s u}{1-u}\right](1-\alpha)k^{\alpha}(1-u)^{-\alpha}\tilde{h}^{-\alpha}}{\bar{B}\phi u^{\phi-1}\tilde{h}^{\phi+\theta-1}} = \frac{\mu}{\lambda}$$

- So

$$\frac{\left[1-\alpha\tau_K-(1-\alpha)\tau_L - s - \frac{\alpha s u}{1-u}\right](1-\alpha)\left(\frac{K}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}+1}L}\right)^{\alpha}(1-u)^{-\alpha}\left(\frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}}\right)^{-\alpha}}{\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}\phi u^{\phi-1}\left(\frac{h}{A^{\frac{g_B}{g_A}\frac{1}{1-\phi-\theta}}}\right)^{\phi+\theta-1}} = \frac{\mu}{\lambda}$$

$$\text{So} \quad \frac{\left[1-\alpha\tau_K-(1-\alpha)\tau_L - s - \frac{\alpha s u}{1-u}\right](1-\alpha)K^{\alpha}\left[A(1-u)hL\right]^{-\alpha}}{\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}A^{\frac{g_B}{g_A}}\phi u^{\phi-1}h^{\phi+\theta-1}} = \frac{\mu}{\lambda}$$

- So

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L - s - \frac{\alpha s u}{1-u}\right](1-\alpha)K^{\alpha}\left[A(1-u)hL\right]^{-\alpha} = \frac{\mu}{\lambda}\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}A^{\frac{g_B}{g_A}}\phi u^{\phi-1}h^{\phi+\theta-1}$$

- So

$$\left[1\underbrace{-\alpha\tau_K-(1-\alpha)\tau_L}_{\text{lost due to taxes}} - \underbrace{\frac{\alpha s u}{1-u}}_{\text{(a)}}\right]\underbrace{(1-\alpha)K^{\alpha}\left[A(1-u)hL\right]^{-\alpha}}_{=w} = \frac{\mu}{\lambda}\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}A^{\frac{g_B}{g_A}}\phi u^{\phi-1}h^{\phi+\theta-1} + sw$$

- (a): Lost due to distortions resulting in physical capital income due to subsidy.
- Recall that w_t is the wage rate per *effective* human capital. That is, 1 unit of human capital earns $A_t w_t$. To interpret this equation with human capital, I multiply both sides by A to have

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L - \frac{\alpha s u}{1-u}\right]A\underbrace{(1-\alpha)K^{\alpha}\left[A(1-u)hL\right]^{-\alpha}}_{=w} = \underbrace{\frac{\mu}{\lambda}\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}A^{\frac{g_B}{g_A}+1}\phi u^{\phi-1}h^{\phi+\theta-1}}_{\text{(b)}} + \underbrace{sAw}_{\text{(c)}}$$

- LHS: net MPH (after taxation/subsidization) in goods production, in units of Y.
- (c) of RHS: subsidy when additional 1 unit of H is used for H production, in units of Y.
- (b) of RHS: Recall that the MPH in human capital production is $B\phi u^{\phi-1}h^{\phi+\theta-1}$. I can rewrite (b) as

$$\frac{\mu}{\lambda}\frac{B(0)}{A(0)^{\frac{g_B}{g_A}}}A^{\frac{g_B}{g_A}+1}\phi u^{\phi-1}h^{\phi+\theta-1} = \underbrace{\frac{\mu}{\lambda}\frac{B(0)}{B}\left(\frac{A}{A(0)}\right)^{\frac{g_B}{g_A}}}_{\text{(d)}} A \underbrace{B\phi u^{\phi-1}h^{\phi+\theta-1}}_{\text{MPH in H production}}$$

- But what is (d)?:

$$\frac{B(0)}{B}\left(\frac{A}{A(0)}\right)^{\frac{g_B}{g_A}} = \frac{1}{e^{g_B t}}e^{g_A t\frac{g_B}{g_A}} = \frac{1}{e^{g_B t}}e^{g_A t\frac{g_B}{g_A}} = 1.$$

- Rewrite the above as

$$\frac{\mu}{\lambda}A \underbrace{B\phi u^{\phi-1}h^{\phi+\theta-1}}_{\text{MPH in H production}}$$

- Therefore,

$$V \equiv \frac{\mu}{\lambda}A$$

is the unit value of H in units of Y. I have

$$V_t = V(0)e^{g_A t} = V_0 e^{g_A t}.$$

- Now I have

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha s u}{1-u}\right]Aw = VB\phi u^{\phi-1}h^{\phi+\theta-1}+sAw$$

where

$$\begin{aligned} w &= (1-\alpha)K^{\alpha}\left[A(1-u)hL\right]^{-\alpha} \\ &= (1-\alpha)(K/Y)^{\alpha/(1-\alpha)} \end{aligned}$$

since $Y_t/K_t = K_t^{\alpha-1}[A_t(1-u_t)h_tL_t]^{1-\alpha}$, so $(K/Y) = K_t^{1-\alpha}[A_t(1-u_t)h_tL_t]^{\alpha-1}$, so $K_t^{\alpha}\left[A_t(1-u_t)h_tL_t\right]^{-\alpha} = (K/Y)^{\alpha/(1-\alpha)}$.

- So

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}\right]A(1-\alpha)(K/Y)^{\alpha/(1-\alpha)}$$
$$=VB\phi u^{\phi-1}h^{\phi+\theta-1}+sA(1-\alpha)(K/Y)^{\alpha/(1-\alpha)}$$

- Now write everything (as in (3"")). At period 0, and assuming $A_0=1$ without loss of generality,

$$\left[1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s\right](1-\alpha)(K/Y)^{\alpha/(1-\alpha)}=V_0B_0\phi u^{\phi-1}h_0^{\phi+\theta-1}$$

so

$$\begin{aligned}
(3\text{"'})\ \rho+\sigma(g_C-g_L) &= \left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]r-\delta_K \\
&= \left[1-\alpha\tau_K-(1-\alpha)\tau_L+\frac{(1-\alpha)su}{1-u}\right]\frac{\alpha}{K/Y}-\delta_K \\
&= \frac{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}}{1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}-s}\phi u^{\phi-1}B_0h_0^{\phi+\theta-1}+g_A-\delta_h \\
&= \left[1-\alpha\tau_K-(1-\alpha)\tau_L-\frac{\alpha su}{1-u}\right]\frac{(1-\alpha)(K/Y)^{\alpha/(1-\alpha)}}{V_0}+g_A-\delta_h
\end{aligned}$$

That is,

$$\begin{aligned}
(\text{a function of consumption growth}) &= (\text{net interest rate})-\delta_K \\
&= (\text{net MPK in Y production})-\delta_K \\
&= (\text{net MPH in H production})+g_A-\delta_h \\
&= (\text{net MPH in Y production})+g_A-\delta_h
\end{aligned}$$

- **Part (3-4)/4 (Regarding (A) and (B'))**: Finally (A) and (B'):

$$(\text{A})\quad \left[d+\frac{u}{1-u}s(1-\alpha)\right]k^{\alpha}[(1-u)\tilde{h}]^{1-\alpha}-(1+\tau_C)c-ek=0$$
$$(\text{B'})\quad \bar{B}u^{\phi}\tilde{h}^{\phi+\theta-1}-f=0$$

- For calibration, it is convenient to write them as

$$\dot{K}=I-\delta_K K$$
$$\text{So (A')}\ g_K=\frac{Y}{K}\frac{I}{Y}-\delta_K=\frac{i}{K/Y}-\delta_K$$

This is eventually the same as (A).

- Similarly,

$$(B)\ \ \dot{h} = B\left[uh\right]^{\phi} h^{\theta} - (\delta_h + g_L)h.$$
$$\text{So } g_h = Bu^{\phi}h^{\phi+\theta-1} - (\delta_h + g_L).$$

At $t = 0$, it becomes

$$g_h = B_0 u^{\phi} h_0^{\phi+\theta-1} - (\delta_h + g_L)$$

- I showed $g_Y = g_C = g_A + g_h + g_L$. So

$$(B")\ g_C - g_A = u^{\phi} B_0 h_0^{\phi+\theta-1} - \delta_h$$

This is eventually the same as (B').

- I put together all equations:

$$\text{(i) } g_K = \frac{i}{K/Y} - \delta_K$$
$$\text{(ii) } g_C - g_A = u^{\phi}\left[B_0 h_0^{\phi+\theta-1}\right] - \delta_h$$
$$\text{(iii) } \rho + \sigma\left(g_C - g_L\right) = \left[1 - \alpha\tau_K - (1-\alpha)\tau_L + \frac{(1-\alpha)su}{1-u}\right] r - \delta_K$$
$$\text{(iv) } = \left[1 - \alpha\tau_K - (1-\alpha)\tau_L + \frac{(1-\alpha)su}{1-u}\right]\frac{\alpha}{K/Y} - \delta_K$$
$$\text{(v) } = \frac{1 - \alpha\tau_K - (1-\alpha)\tau_L - \frac{\alpha su}{1-u}}{1 - \alpha\tau_K - (1-\alpha)\tau_L - \frac{\alpha su}{1-u} - s}\phi u^{\phi-1} B_0 h_0^{\phi+\theta-1} + g_A - \delta_h$$
$$\text{(vi) } = \left[1 - \alpha\tau_K - (1-\alpha)\tau_L - \frac{\alpha su}{1-u}\right]\frac{(1-\alpha)(K/Y)^{\alpha/(1-\alpha)}}{V_0} + g_A - \delta_h$$

- Notice that for growth rates, I also have

$$\text{(vii) } g_Y = g_C = g_K = g_A + g_h + g_L$$

and

$$\text{(viii) } g_B + (\phi + \theta - 1)g_h = 0$$