Growth Lecture Notes

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Contents

Neoclassical Growth Model: Calibration

References

- Lucas, R. E., Jr. (2011), Macroeconomics: A Short Course, Chapter 5, unpublished manuscript.
- [O] Lucas, R. E., Jr. (2004), "The Industrial Revolution: Past and Future," The Region, May, 5-20.

- Ultimate goal of growth economics: To understand why poor countries are poor.
- Consider a growth model that fits the data.
  (Neoclassical) (US, 1960-2005)
  "Neoclassical" is just a name of a group of models, like "Marxist" or "Keynesian".

1. The Model

- (output) $= F$(inputs): Production function.
  "inputs": labor (human capital), physical capital, ...
- Many economists find the following functional form reasonable. Why? It fits relatively well to data. (We will see.)
- $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$: **Cobb-Douglas production function**.
  - Y_t: total output (measured by real GDP in constant dollars)
  - K_t: physical capital stock, such as buildings, machines, computers, ... (Typically not in data. Sometimes it is in data, but it is constructed from the method that we discuss below anyway.)
  - $0 < \alpha < 1$: a parameter often called "physical capital share" (Why? We will see.)

- L_t: labor input (measured in manhours or #workers)
- A_t: productivity, which is an accumulation of education, R&D, efficiency gains from elimination of monopoly power, ... (not in data)
- $A_t L_t$: "effective units" of labor

- What determines the total output?: K_t, L_t, A_t
- Sometimes we write $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$. A_t in this form is called the "**total factor productivity** (TFP)". These two set-ups are eventually the same.
- $\boxed{L_{t+1} = L_t(1 + g_L)\text{: (exogenous) population growth.}}$
- $\boxed{A_{t+1} = A_t(1 + g_A)\text{: (exogenous) productivity growth.}}$
- These two variables grow *exogenously*, so our model does not explain why they grow.
- But this model cares about how K_t grows. We specify it as follows:
  - $K_{t+1} = I_t + (1 - \delta)K_t$
  - I_t: Investment (observed in data)
  - $Y_t = C_t + I_t$. Out of total output, C_t units are consumed, and the remaining I_t units are invested for future productions.
  - $s_t \equiv I_t / Y_t$ is an investment rate (savings rate).
  - δ: depreciation rate.
- So $\boxed{K_{t+1} = s_t Y_t + (1 - \delta)K_t\text{: accumulation of physical capital (a.k.a. law of motion of } K_t\text{).}}$

2. Calibration

- Calibration is complementary to regression.
  - Both approaches require to make assumptions that can be criticized.
  - Both approaches have their advantages: While a regression approach provides estimates that can be statistically interpreted, a calibration approach can provide a coherent theoretical foundation.
- Two assumptions to conduct calibration:
- (i) Each variable grows at a constant rate.

- This assumption is often called a "**balanced growth path**" (BGP).
- This is reasonable. The growth rate of any variable doesn't rise or decline forever.
- (ii) s_t is constant. (We will drop the subscript t now.) This is also reasonable from data.
- $\boxed{\text{Data: } s = 0.20.}$ (average of 1960-2005. For each year t, divide gross domestic investment, in NIPA Table 1.1.5, by GDP, in NIPA Table 1.1.)
- There is a cross-country variation on s. But the U.S. time-series is stable.
- $\boxed{\text{Data: } g_L = 1.1\%.}$ per year. (average of 1960-2005, NIPA Table 7.1.)
- $\boxed{\text{Data: } g_Y = 3.3\%.}$ per year. (average of 1960-2005, NIPA Table 7.1. This is the sum of g_L and "GDP per capita in chained dollars".)
- **(1) Calibration of g_A**
- We investigate the LOM of K:

$$K_{t+1} = sK_t^{\alpha}(A_tL_t)^{1-\alpha} + (1-\delta)K_t.$$

$$\text{So } \frac{K_{t+1}}{K_t}\frac{K_t}{A_tL_t} = s\left(\frac{K_t}{A_tL_t}\right)^{\alpha} + (1-\delta)\frac{K_t}{A_tL_t}.$$

$$\text{So } (g_K + \delta)\frac{K_t}{A_tL_t} = s\left(\frac{K_t}{A_tL_t}\right)^{\alpha}.$$

- From the following figure, a strictly positive, constant solution for $\frac{K_t}{A_tL_t}$ uniquely exists.

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|                  +  +
|              ++
|         +  +
|    +   +
| +   +
|+  +
+------------------------- Kt/(AtLt)
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- Since $\frac{K_t}{A_tL_t}$ is constant, $\boxed{(\text{growth rate of } K_t) = (\text{growth rate of } A_tL_t)}$.
- But from the production function: $Y_t = K_t^{\alpha}(A_tL_t)^{1-\alpha}$,

- $\log(Y_t) = \log(K_t^\alpha (A_t L_t)^{1-\alpha}) = \alpha \log(K_t) + (1-\alpha)\log(A_t L_t)$.
- So $\underbrace{\log(Y_{t+1}) - \log(Y_t)}_{g_Y} = \alpha \underbrace{[\log(K_{t+1}) - \log(K_t)]}_{\text{(growth rate of } K_t)} + (1-\alpha) \underbrace{[\log(A_{t+1}L_{t+1}) - \log(A_t L_t)]}_{\text{(growth rate of } A_t L_t)}$.
- Note: Growth rate of X_t is (i) $X_{t+1}/X_t - 1$ or (ii) $\log(X_{t+1}) - \log(X_t)$.

- Hence, this result implies that

$$g_Y = \text{(growth rate of } A_t L_t).$$
$$\text{So } g_Y = g_A + g_L.$$

- $\boxed{\text{Result: } g_A = g_Y - g_L = 3.3\% - 1.1\% = 2.2\%.}$
- **(2) Calibration of δ**
- We have

$$K_{t+1} = sY_t + (1-\delta)K_t.$$
$$\frac{K_{t+1}}{K_t}\frac{K_t}{Y_t} = s + (1-\delta)\frac{K_t}{Y_t}.$$
$$\text{So } (1+g_K)\frac{K_t}{Y_t} = s + (1-\delta)\frac{K_t}{Y_t}.$$
$$\text{So } \underbrace{g_K}_{0.033}\frac{K_t}{Y_t} = \underbrace{s}_{0.2} - \delta\frac{K_t}{Y_t}.$$

- Note: $g_K = 0.033$ since we showed $g_K = g_Y$.
- $\boxed{\text{Data: } \delta\frac{K_t}{Y_t} = 0.12.}$ (average of 1960-2005, NIPA Table 1.1. This is a ratio of "consumption of fixed capital" to Y.)
- So

$$0.033\frac{K_t}{Y_t} = 0.20 - 0.12.$$

- Hence, $\boxed{\text{Result: } \frac{K_t}{Y_t} = 2.42.}$
- Now we measure the value of all U.S. physical capital.
- Also, $\boxed{\text{Result: } \delta = 0.05.}$
- Another cool result: 5% of physical capital stock depreciates per year.
- **(3) Calibration of α**

- (interest rate) = (marginal product of physical capital).
- $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} \Longrightarrow r_t \equiv \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha}$.
- (phyiscal capital income) = $r_t K_t = \alpha K_t^{\alpha}(A_t L_t)^{1-\alpha} = \alpha Y_t$
- Similarly, (labor income) = $w_t A_t L_t = (1-\alpha) Y_t$
- Data: (income share of K) = 0.32. (NIPA Table 1.12. This is a fraction of "compensation of employees" out of "GDP" minus "propietors' income" minus "taxes on production and imports".)
- Result: $\alpha = 0.32$.
- We can also compute the interest rate. Since $r_t K_t = \alpha Y_t$,

$$r_t = \frac{\alpha}{K/Y} = \frac{0.32}{2.42} = 13.2\%$$

- Result: $r = 13.2\%$.
- 1 unit of apple –(1year)——> $1-\delta$ units
- $+ r$ units (as a compensation)
- $r - \delta$: net interest rate, $13.2\% - 5.0\% = 8.2\%$.
- **(4) Experiments on** $\{K_t\}$
- We already have $\frac{K_t}{Y_t} = 2.42$.
- Assume in 2008, the economy is in a steady state:
  - Real Per-capita GDP (Y_{2008}/L_{2008}) = \$45,000 (data)
  - Population (L_{2008}) = 200 million (data)
  - Real GDP (Y_{2008}) = \$45,000 x 200 million
  - Physical Capital Stock (K_{2008}) = We know $K/Y = 2.42$. So $K_{2008} = 2.42 Y_{2008} =?$

    (We measured something not observed!)
- Example: What happens if the investment rate (s) increases to 0.3 permanently?
- Use $K_{2009} = \underbrace{0.3}_{\text{instead of } s=0.2} Y_{2008} + (1-\delta)K_{2008}$. Continue for all future years.

- Example: What happens if the investment rate is still 0.2, but there is an FDI inflow so K_{2009} increased by \$100 billion?

- $K_{2009} = 0.2Y_{2008} + (1-\delta)K_{2008} + \underbrace{(100 \text{ billion})}_{\text{temporary increase in } K_t}$

3. Growth Decomposition

- The ultimate question. What are the sources of the U.S. per-capita income growth?

- $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$

  $\Longrightarrow\ Y_t/L_t = (K_t/L_t)^\alpha (A_t)^{1-\alpha}$

  $\Longrightarrow\ \log(Y_t/L_t) = \alpha \log(K_t/L_t) + (1-\alpha)\log A_t$

  $\Longrightarrow\ \underbrace{g_{Y/L}}_{(A)} = \underbrace{\alpha g_{K/L}}_{(B)} + \underbrace{(1-\alpha)g_A}_{(C)}$

  where (A): Real per-capita GDP growth, (B): the contribution from (K_t/L_t)'s accumulation, (C) : the contribution from A_t growth

- (A) = $g_Y - g_L = 3.3\% - 1.1\% = 2.2\%$

  (B) = $\alpha(g_K - g_L) = \alpha(g_Y - g_L) = 0.32 \times 2.2\%$

  (C) = (A)-(B) = $0.68 \times 2.2\%$

  (B)/(A) = 32% (32% of per-capita GDP growth is due to per-capita K accumulation.)

  (C)/(A) = 68% (68% of it is due to exogenous productivity growth.)

- This method is no longer popular now.

- Why? If A rises $\longrightarrow$ Y/L rises $\longrightarrow$ K/L rises since K/Y is constant.

  That is, K/L may rise eventually because A rises!

- Need something not affected by A.

- Use K/Y instead of K/L to represent physical capital accumulation. (Hall and Jones (1999).)

- $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$

  $\Longrightarrow\ Y_t^{1-\alpha} = (K_t/Y_t)^\alpha (A_t L_t)^{1-\alpha}$

  $\Longrightarrow\ (Y_t/L_t)^{1-\alpha} = (K_t/Y_t)^\alpha A_t^{1-\alpha}$

$$\implies Y_t/L_t = (K_t/Y_t)^{\alpha/(1-\alpha)} A_t$$

$$\implies \underbrace{g_{Y/L}}_{(A)} = \underbrace{\frac{\alpha}{1-\alpha} g_{K/Y}}_{(B')} + \underbrace{g_A}_{(C')}$$

- But (B') is 0.
- So $(C')/(A) = 100\%$! Everything is productivity growth.
- Conclusion: Whichever accounting we follow, the contribution from productivity growth is large (either 68% or 100%).
- Then what is this productivity growth? Need a better model with human capital, R&D, etc., which can explain what A_t is.

4. Introduction of the Consumers

- We have assumed that s_t is constant.
- Consumers make decisions on s_t. Extend the model so that investment rate is **endogenously determined**.
- $u(C_t/L_t)$: The utility function for each (identical) consumer.
- $u(C_t/L_t) = (C_t/L_t)^{1-\sigma}/(1-\sigma)$, $\sigma > 0$ and $\sigma \neq 1$. This is called a **CRRA** (constant relative risk aversion) utility function.

  (This becomes the log utility, $u(C_t/L_t) = \log(C_t/L_t)$, if σ converges to 1.)
- There are L_t consumers. We typically assume that $u(C_t/L_t)L_t$ is maximized by the **representative consumer** (who represent the preferences of all the people):

$$\max_{\{C_t, I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t$$

s.t.

$$\begin{aligned} K_{t+1} &= I_t + (1-\delta)K_t, \ K_0 \text{ given}, \\ Y_t &= C_t + I_t, \\ Y_t &= K_t^{\alpha}(A_t L_t)^{1-\alpha}, \\ L_{t+1} &= L_t(1+g_L), \ L_0 \text{ given}, \\ A_{t+1} &= A_t(1+g_A), \ A_0 \text{ given}. \end{aligned}$$

- Eliminate Y_t and I_t :

$$K_{t+1} = K_t^\alpha (A_t L_t)^{1-\alpha} - C_t + (1-\delta)K_t.$$

- Now the problem:

$$\max_{\{C_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t$$

s.t.

$$\begin{aligned} K_{t+1} &= K_t^\alpha (A_t L_t)^{1-\alpha} - C_t + (1-\delta)K_t, \;\; K_0 \text{ given}, \\ L_{t+1} &= L_t(1+g_L), \;\; L_0 \text{ given}, \\ A_{t+1} &= A_t(1+g_A), \;\; A_0 \text{ given}. \end{aligned}$$

- Now the problem:

$$\underbrace{\max_{\{K_{t+1}\}_{t=0}^\infty}}_{K_1, K_2, \ldots} \sum_{t=0}^\infty \beta^t \frac{(\overbrace{K_t^\alpha (A_t L_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}}^{=C_t})^{1-\sigma}}{1-\sigma} \left(\frac{1}{L_t}\right)^{-\sigma}$$

s.t.

$$\begin{aligned} L_{t+1} &= L_t(1+g_L), \;\; L_0 \text{ given}, \\ A_{t+1} &= A_t(1+g_A), \;\; A_0 \text{ given}, \end{aligned}$$

and K_0 given.

- Rewriting,

$$\begin{aligned} &\max_{K_1, K_2, \ldots} \frac{(K_0^\alpha (A_0 L_0)^{1-\alpha} + (1-\delta)K_0 - K_1)^{1-\sigma}}{1-\sigma} \left(\frac{1}{L_0}\right)^{-\sigma} \\ &+ \beta \frac{(K_1^\alpha (A_1 L_1)^{1-\alpha} + (1-\delta)K_1 - K_2)^{1-\sigma}}{1-\sigma} \left(\frac{1}{L_1}\right)^{-\sigma} \\ &+ \ldots \\ &+ \beta^t \frac{(K_t^\alpha (A_t L_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1})^{1-\sigma}}{1-\sigma} \left(\frac{1}{L_t}\right)^{-\sigma} \\ &+ \beta^{t+1} \frac{(K_{t+1}^\alpha (A_{t+1} L_{t+1})^{1-\alpha} + (1-\delta)K_{t+1} - K_{t+2})^{1-\sigma}}{1-\sigma} \left(\frac{1}{L_{t+1}}\right)^{-\sigma} \\ &+ \ldots \end{aligned}$$

- So the FOC wrt K_{t+1} is

$$-\beta^t C_t^{-\sigma}\left(\frac{1}{L_t}\right)^{-\sigma} + \beta^{t+1}C_{t+1}^{-\sigma}(\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + 1 - \delta)\left(\frac{1}{L_{t+1}}\right)^{-\sigma} = 0.$$

So

$$1 = \beta\left(\frac{C_{t+1}/L_{t+1}}{C_t/L_t}\right)^{-\sigma}(\underbrace{\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}}_{=r_{t+1}\text{ (interest rate bw } t \text{ and } t+1)} + 1 - \delta)$$

- This equation is often called the **Euler equation**. The Euler equation relates the consumption growth ($\frac{C_{t+1}/L_{t+1}}{C_t/L_t}$) to the interest rate.
- So what is the difference between this version, with the representative consumer, and the previous version without it?
  - Now a new equation, Euler equation, is added.
  - Now two more parameters, β and σ, are added.
  - The second assumption that s_t is constant is no longer required. It arises naturally because (i) the above Euler equation implies that $\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}$ is constant (since $\frac{C_{t+1}/L_{t+1}}{C_t/L_t}$ is constant on the BGP), (ii) but from $Y_t = K_t^\alpha(A_tL_t)^{1-\alpha}$, this implies $g_Y = g_K$, (iii) also, $K_{t+1} = I_t + (1-\delta)K_t \implies K_{t+1}/K_t = I_t/K_t + (1-\delta)$, so $g_Y = g_K = g_I$ on BGP. Then s_t is constant since $g_Y = g_I$.
- Nothing else. All the previous calibration results hold. We simply have an additional equation w/ two unknowns, β and σ.
- $1 = \beta(1 + 0.033 - 0.011)^{-\sigma}(0.132 + 1 - 0.05)$.
- If $\sigma = 1$ (log utility), then we can calibrate $\beta = 0.95$.
- Now we have the quantified utility function. We can analyze how people will react, e.g., to a tax reform.

EXERCISES

1. True or False? Explain: The current U.S. investment rate is 20% or lower, while China's level is as high as 40%. This implies that if the U.S. can raise the investment rate to 40%, then the U.S. GDP growth rate will converge to China's GDP growth rate.

   Answer: *FALSE - The neoclassical growth model (we discussed in class) implies that long-run growth rate on the balanced growth path does not depend on the investment rate itself:*

$$\underbrace{g_{Y/L}}_{(A)} = \underbrace{\frac{\alpha}{1-\alpha} g_{K/Y}}_{(B')} + \underbrace{g_A}_{(C')}$$

   Here, $(B) = 0$ as long as the investment rate is constant, so $g_{Y/L} = g_A$! In other words, a rise in investment rate will only temporarily raise the growth rate, and long-run growth will converge to productivity growth.

2. Visit the Penn World Table website at http://pwt.econ.upenn.edu/php_site/pwt_index.php. Pick up your favorite country except the United States. Call this country A. Assume the model considered in class:

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t L_t)^{1-\alpha}, \\ L_{t+1} &= L_t(1+g_L), \\ A_{t+1} &= A_t(1+g_A), \\ K_{t+1} &= s_t Y_t + (1-\delta)K_t, \end{aligned}$$

   for all year t, where $\alpha = 0.32$ and $\delta = 0.05$. All notations are the same as in class.

   (a) Obtain the data on $\{L_t\}$. (I suggest you to use "POP".) Suppose the starting year of the data is τ_1 (e.g., 1950) and the ending year is τ_2 (e.g., 2007). Obtain the average annual growth rate of $\{L_t\}$ between τ_1 and τ_2. Use this as an estimate for g_L. What is your prediction on L_{2015}?

   (b) Obtain the data on $\{Y_t/L_t\}$. (I suggest you to use "rgdpl".) Obtain the average annual growth rate of $\{Y_t\}$ between τ_1 and τ_2.

   (c) Obtain the data on $\{s_t\}$. (I suggest you to use "ki".) Plot the evolution of $\{s_t\}$ between τ_1 and τ_2. Is it stable? Is it growing or shrinking? (Feel free to add your own explanation on this evolution of $\{s_t\}$ in country A.) Obtain the average of $\{s_t\}$.

(d) Now we want to measure $\{K_t\}$ (which is not in data). To begin with, assume a balanced growth path in which (i) all variables grow at constant rates and (ii) s_t is constant. Dividing both sides of the last equation by K_t, we have $K_{t+1}/K_t = s_t Y_t/K_t + (1-\delta)$. What is a constant level of K_t/Y_t on this balanced growth path?

(e) Now assume that the value of K/Y that you obtained holds for the middle year, $(\tau_1 + \tau_2)/2$ (or some year close to it, chosen at your discretion). Obtain $K_{(\tau_1+\tau_2)/2}$ for this year.

(f) Now we obtain all remaining K_t's. To deal (at least partly) with the problem that country A may not be on the balanced growth path, do not just apply your K/Y to all Y_t's to find K_t's. Rather, use the law of motion of K_t, the data on s_t and Y_t, and $\delta = 0.05$, to obtain all remaining K_t's from τ_1 to τ_2. (For example, you can obtain $K_{(\tau_1+\tau_2)/2-1}$ from $K_{(\tau_1+\tau_2)/2} = s_{(\tau_1+\tau_2)/2-1}Y_{(\tau_1+\tau_2)/2-1} + (1-\delta)K_{(\tau_1+\tau_2)/2-1}$ where the only unknown is $K_{(\tau_1+\tau_2)/2-1}$.)

(g) Measure $\{A_t\}$ for all years, from τ_1 to τ_2. What is your prediction on A_{2015}?

(h) Now for the entire time horizon τ_1 through τ_2, discuss what fractions of per-capita output growth in country A are due to the accumulation in K_t/Y_t and to the growth of A_t, respectively. Was the physical capital accumulation important for this country's economic growth? Explain.

(i) Assume that from year $\tau_2 + 1$ on, country A has the same growth rates of A_t and L_t and the same level of s_t as in averages of years $\tau_2 - 9$ through τ_2. What are your predictions on K_{2015} and Y_{2015}? On K_{2050} and Y_{2055}? What are the growth rates of K_t and Y_t between 2049 and 2050?

(j) Suppose from the year $\tau_2 + 1$, the investment rate has risen to $1.5\overline{s}$ (by 50% from the previous average) and will stay at this new level forever. Assume the growth rate of A_t will be the same, forever, as the average growth of A_t from τ_1 to τ_2. What is your prediction on the growth of Y_t/L_t for the next 100 years? (Show by figure or whatever you believe is the best so that country A's government officials without PhDs can understand.) If country A's investment rate permanently rises, how will it affect the short-run and long-run per-capita output growth? Explain.

(k) Disregard (j) and consider (i). Now assume that there is a one-time foreign direct investment of 100 billion dollars into country A in year $\tau_2 + 1$. What are your predictions on K_{2015} and Y_{2015}? On K_{2050} and Y_{2055}? What are the growth rates of K_t and Y_t between 2049 and 2050?

3. Consider the U.S. economy. Suppose that the government wants to tax more heavily on the rich. Since the rich tend to have more financial assets, the government has

decided to raise the tax rate on physical capital income. But every policy has a side effect, and the government is concerned about what will happen to long-run growth of the U.S. economy.

To address this issue seriously, we have to introduce all types of taxes (such as labor income tax, property tax, ...) and subsidies (to education, health care, ...). In this exercise, assume that the tax on physical capital income is the only type of taxation. The government throws away all tax revenue to the ocean. (Or we may assume that all tax revenue is lump-sum-transfered to the representative consumer. But this alternative assumption does not change any results.) To be specific, we consider the following model. The representative consumer solves

$$\max_{\{C_t,I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(C_t/L_t) \times L_t,\ 0<\beta<1$$

s.t.

$$\begin{aligned}
K_{t+1} &= I_t + (1-\delta)K_t,\ 0<\delta<1,\ K_0 \text{ given},\\
Y_t &= C_t + I_t + \tau\alpha Y_t,\ 0<\tau<1,\\
Y_t &= K_t^{\alpha}(A_tL_t)^{1-\alpha},\ 0<\alpha<1,\\
L_{t+1} &= L_t(1+g_L),\ L_0 \text{ given},\\
A_{t+1} &= A_t(1+g_A),\ A_0 \text{ given}.
\end{aligned}$$

Notice that $\tau\alpha Y_t$ is the amount of tax paid to the government. τ is a flat tax rate on physical capital income, αY_t. The representative consumer takes τ as exogenous (because it is the government that controls it). All notations are the same as in class. Notice that I further simplified the set-up by assuming the log utility (i.e., $\sigma = 1$).

(a) Calibrate δ, α, g_A and β in the model, assuming a balanced growth path (in which all variables grow at constant rates). Use the following observations:

- $\tau = 36\%$: McGrattan and Prescott (Federal Reserve Bank of Minneapolis Quarterly Review, 2000), Table 2.
- $s \equiv I_t/Y_t = 0.20$: 1960-2005, NIPA Table 1.1.5.
- $g_L = 1.1\%$: 1960-2005, NIPA Table 7.1.
- $g_Y = 3.3\%$: 1960-2005, NIPA Table 7.1.
- $\delta K/Y = 0.12$: 1960-2005, NIPA Table 1.1.
- (before-tax income share of K) $= 0.32$: 1960-2005, NIPA Table 1.12.

Answer: *First, the law of motion of physical capital is not changed by introduction of τ. So as in class, $K/Y = 2.42$ and $\delta = 0.05$. Second, the payment to physical capital income is the same as in class. So $\alpha = 0.32$. Third, we still have $g_A = 2.2\%$.*

The representative consumer's problem is

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log\left(\frac{(1-\tau\alpha)K_t^{\alpha}(A_tL_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}}{L_t}\right) \times L_t$$

s.t.

$$\begin{aligned} L_{t+1} &= L_t(1+g_L),\ L_0 \text{ given}, \\ A_{t+1} &= A_t(1+g_A),\ A_0 \text{ given}, \end{aligned}$$

and K_0 given. The first-order condition is

$$-\beta^t \frac{1}{C_t} L_t + \beta^{t+1} \frac{(1-\tau\alpha)\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + (1-\delta)}{C_{t+1}} L_{t+1} = 0.$$

So

$$1 = \beta \frac{C_t/L_t}{C_{t+1}/L_{t+1}} \left[(1-\tau\alpha)\frac{\alpha}{K/Y} + (1-\delta)\right].$$

So

$$\begin{aligned} \beta &= \frac{1}{(1-\tau\alpha)\frac{\alpha}{K/Y} + (1-\delta)}(1+g_C-g_L) \\ &= \frac{1}{(1-0.36\times 0.32)\times 0.32/2.42 + (1-0.05)}(1.022) \\ &= 0.960 \end{aligned}$$

(b) Suppose that in January 2011, the government permanently raises the tax rate τ from 36% to 50%. After several years, the economy has reached a new balanced growth path (in which all variables grow at constant rates). Discuss what happens to the level of per-worker GDP and the growth rate of per-worker GDP. Provide numbers: What would be the level of Y_t/L_t on the new balanced growth path under $\tau = 50\%$ (Scenario 2), compared to the level on the old balanced growth path if $\tau = 36\%$ were maintained (Scenario 1)? Draw the time-series paths of Y_t/L_t under these two scenarios.

Hint: All parameter values, including g_A and g_L, are exogenously given. They do not change no matter what happens to τ. But the consumer's decision may be affected, causing the levels of, e.g., s and K/Y, to change. To think about what will happen to the level of Y_t/L_t, try to use $Y_t/L_t = (K/Y)^{\alpha/(1-\alpha)} A_t$.

Answer: *Rewrite the conditions under the balanced growth path:*

$$
\begin{aligned}
g_Y &= g_A + g_L, \\
1 &= \beta \frac{1}{1+g_Y-g_L}\left[(1-\tau\alpha)\frac{\alpha}{K/Y} + (1-\delta)\right], \\
s &= (g_Y+\delta)(K/Y).
\end{aligned}
$$

With exogenously given parameter values, $g_A = 2.2\%$, $g_L = 1.1\%$, $\beta = 0.96$, $\tau = 0.5$, $\alpha = 0.32$, and $\delta = 0.05$, *the representative consumer makes her decisions on* C_t *and* I_t*:*

$$
\begin{aligned}
g_Y &= 0.022 + 0.011, \\
1 &= 0.96 \frac{1}{1+g_Y-0.011}\left[(1-0.5\times 0.32)\frac{0.32}{K/Y} + (1-0.05)\right], \\
s &= (g_Y+0.05)(K/Y).
\end{aligned}
$$

This gives

$$
\begin{aligned}
g_Y &= 0.033, \\
K/Y &= \frac{1}{\left(\frac{1.022}{0.96} - 0.95\right)\frac{1}{0.32\times(1-0.5\times 0.32)}} = 2.35, \\
s &= (0.033+0.05)2.35 = 0.195.
\end{aligned}
$$

Hence, g_Y *is the same as before.* K/Y *and* s *are slightly lower than before. Then what would be the level of* Y_t*? Notice that*

$$
\begin{aligned}
Y_t &= K_t^{\alpha}(A_tL_t)^{1-\alpha}. \\
\text{So } Y_t^{1-\alpha} &= \left(\frac{K_t}{Y_t}\right)^{\alpha}(A_tL_t)^{1-\alpha}. \\
\text{So } Y_t &= (K/Y)^{\alpha/(1-\alpha)}(A_tL_t).
\end{aligned}
$$

Under $\tau = 0.36$*, we had* $K/Y = 2.42$, so $Y_t/L_t = 2.42^{0.32/(1-0.32)}A_t = 1.516A_t$. *On the other hand, under* $\tau = 0.50$*, we now have* $K/Y = 2.35$*. So* $Y_t/L_t = 2.35^{0.32/(1-0.32)}A_t = 1.495A_t$*. Note that* A_t *is the same between these two* τ *values.*

Conclusion: Under the new policy, the per-capita income on the balanced growth path will be $1.495/1.516 = 98.6\%$ *compared to the level that would be attained if the old policy continued. But the growth rate of per-capita income is the same.*

(c) If the physical income tax is completely eliminated (i.e., if τ becomes 0), then what will happen to the level of Y_t/L_t on the new balanced growth path (Scenario 3), compared to the old one if $\tau = 36\%$ were maintained (Scenario 1)? To the growth rate of Y_t/L_t?

Answer: *Rewrite the conditions under the balanced growth path:*

$$\begin{aligned} g_Y &= g_A + g_L, \\ 1 &= \beta \frac{1}{1+g_Y-g_L}\left[(1-\tau\alpha)\frac{\alpha}{K/Y} + (1-\delta)\right]. \end{aligned}$$

With exogenously given parameter values,

$$\begin{aligned} g_Y &= 0.022 + 0.011, \\ 1 &= 0.96 \frac{1}{1+g_Y-0.011}\left[\frac{0.32}{K/Y} + (1-0.05)\right]. \end{aligned}$$

This gives

$$\begin{aligned} g_Y &= 0.033, \\ K/Y &= \frac{1}{\left(\frac{1.022}{0.96}-0.95\right)\frac{1}{0.32}} = 2.79. \end{aligned}$$

Under $\tau = 0.36$, we had $K/Y = 2.42$. So $Y_t/L_t = 2.42^{0.32/(1-0.32)}A_t = 1.516A_t$. On the other hand, under $\tau = 0$, we will have $K/Y = 2.79$. So $Y_t/L_t = 2.79^{0.32/(1-0.32)}A_t = 1.621A_t$.

Conclusion: If physical capital tax is eliminated, the per-capita output on the balanced growth path will be 7% higher ($1.621/1.516 = 1.07$), compared to the level that would be under the old policy. The growth rate is the same.

4. The representative consumer solves

$$\max_{\{C_t,I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} \times L_t$$

for $0 < \beta < 1$, $\sigma > 0$, $\sigma \neq 1$, subject to

$$\begin{aligned} K_{t+1} &= I_t + (1-\delta)K_t,\ 0<\delta<1,\ K_0 \text{ given}, \\ Y_t &= (1+\tau)C_t + I_t,\ 0<\tau<1, \\ Y_t &= K_t^{\alpha}(A_tL_t)^{1-\alpha},\ 0<\alpha<1, \\ L_{t+1} &= L_t(1+g_L),\ L_0 \text{ given},\ g_L>0, \\ A_{t+1} &= A_t(1+g_A),\ A_0 \text{ given},\ g_A>0. \end{aligned}$$

Here, C_t is consumption (at period t), L_t is population, K_t is physical capital, I_t is the investment in physical capital, Y_t is the output, and A_t is exogenously growing productivity. Notice that τ is a flat tax rate on consumption. The representative consumer takes τ as exogenous. The government throws away all tax revenues to the ocean.

Discuss how the consumption tax affects the growth rates of output and physical capital on a balanced growth path in which all variables grow at constant rates. That is, if τ increases, then after a new balanced growth path is reached, do the growth rates of Y_t and K_t increase or decrease? Describe your results in plain English. (It is sufficient to compare the old and new balanced growth paths.)

Answer: *Eliminate the constraints as follows. The fourth and fifth constraints are about exogenous variables, so we can leave them for now. Eliminate Y_t in the second and third constraints to have*

$$I_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} - (1+\tau)C_t.$$

Plugging this into the first equation to have

$$K_{t+1} = K_t^{\alpha}(A_t L_t)^{1-\alpha} - (1+\tau)C_t + (1-\delta)K_t,$$
$$\text{So } C_t = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}}{1+\tau}.$$

The representative consumer's problem is

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}}{(1+\tau)L_t}\right)^{1-\sigma}}{1-\sigma} \times L_t$$

and two constraints, $L_{t+1} = L_t(1+g_L)$ and $A_{t+1} = A_t(1+g_A)$. The FOC is

$$\beta^t \left(\frac{C_t}{L_t}\right)^{-\sigma} \frac{L_t}{(1+\tau)L_t} = \beta^{t+1} \left(\frac{C_{t+1}}{L_{t+1}}\right)^{-\sigma} \frac{L_{t+1}}{(1+\tau)L_{t+1}} [\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + 1 - \delta]$$

Rearranging,

$$1 = \beta \left(\frac{C_{t+1}/L_{t+1}}{C_t/L_t}\right)^{-\sigma} [\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha} + 1 - \delta]$$

Notice that τ cancels out. On a balanced growth path in which all variables grow at constant rates, this first order condition implies that $g_K = g_A + g_L$ since all terms except $\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}$ are constant. Since $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, we have $g_Y = g_K = g_A + g_L$. But then, g_Y and g_K are equal to an exogenous growth rate, $g_A + g_L$, which is not affected by τ or any choice variable. Hence, τ does not affect the growth rates of Y_t and K_t on the balanced growth path.

Neoclassical Growth Model: Continuous-Time Representation

References

- Barro, R. J., and X. Sala-i-Martin (2004), Economic Growth, 2nd Edition, MIT Press, Cambridge, MA, Chapter 1 and Appendix A.3.

- Many economists use the continuous-time representation.

1. Notation

- $\dot{x}(t) \equiv dx(t)/dt$

```
x(t)|
    |         *
    |      *
    |   *
    |
    +-------------t
```

  $\dot{x}(t)$ is the slope at a given level of t.

- Note: In this lecture note, "(t)" (t in parenthesis) implies that the time is continuous. "$_t$" (t as subscript) implies that the time is discrete.
- So if $\dot{x}(0) = 2$, then around $t = 0$, $x(t)$ increases by 2 in 1 period.
- Roughly, $\dot{x}(t)$ corresponds to $x_{t+1} - x_t$ in a discrete-time set-up.
- So $\dot{x}(t)/x(t)$ is the growth rate of $x(t)$ around t.

- **Example 1**: $K_{t+1} = I_t + (1-\delta)K_t \Longrightarrow K_{t+1} - K_t = I_t - \delta K_t$ (discrete time)
  $\Longrightarrow \boxed{\dot{K}(t) = I(t) - \delta K(t)}$ (continuous time)
  In year 2008, $\dot{K}(2008) = I(2008) - \delta K(2008)$
  : Around 1 July 2008, by how much K increases in one year is the investment in 2008 minus a fraction δ of K.

- **Example 2**: $L_{t+1} = L_t(1+g_L) \Longrightarrow (L_{t+1} - L_t)/L_t = g_L$ (discrete time)
  $\Longrightarrow \boxed{\dot{L}(t)/L(t) = g_L}$ (continuous time)
  $\Longrightarrow$ A solution for this differencial equation is $\boxed{L(t) = L(0)e^{g_L t}}$. (You have to remember this solution.)
  (Proof: $\dot{L}(t) = L(0)e^{g_L t}g_L$. So $\dot{L}(t)/L(t) = g_L$.)

- Digresson: From $L(t) = L(0)e^{g_L t}$, we have $\log L(t) = \log L(0) + g_L t$. Hence, $\log L(t+1) - \log L(t) = g_L$. Now we understand the growth rate better:

$$\begin{aligned} g_L &= \frac{L_{t+1}}{L_t} - 1 \quad \text{(discrete-time definition)} \\ &\approx \log L(t+1) - \log L(t) \quad \text{(continuous-time definition)} \end{aligned}$$

- **Example 3**: Similarly, $A_{t+1} = A_t(1+g_A) \Longrightarrow \boxed{\dot{A}(t)/A(t) = g_A}$ and $\boxed{A(t) = A(0)e^{g_A t}}$.

- Now rewrite our original model of economic growth.

- $\boxed{Y(t) = C(t) + I(t)}$ and $\boxed{Y(t) = K(t)^\alpha (A(t)L(t))^{1-\alpha}}$ are the same as before.

- What about $\max_{\{C_t, I_t\}} \sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t$?

- We write $\boxed{\max_{\{C(t), I(t)\}} \int_0^\infty e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t) dt}$.
  (Just as $A_t = A_0(1+g_A)^t$ is replaced by $A(t) = A(0)e^{g_A t}$, $\beta^t \equiv (1-\rho)^t$ is replaced by $e^{-\rho t}$.)

2. Solution to the Determinisitic Continuous-Time Optimization Problem

- A more formal introduction can be found at Barro and Sala-i-Martin (2004), pp. 604-618.

- In general,

$$\max_{\left\{\boxed{u(t)}\right\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(\underbrace{x(t)}_{\text{state variable}}, \underbrace{u(t)}_{\text{control variable}})dt, \quad \rho > 0$$

s.t.

$$\boxed{\dot{x}(t)} = g(x(t), u(t)), \ x(0) \text{ given.}$$

- Just as we set up the Lagrangian function, set up the **Hamiltonian function**:

$$H(x(t), u(t), \lambda(t)) = h(x(t), u(t)) + \underbrace{\lambda(t)}_{\text{co-state variable}} g(x(t), u(t)).$$

- Then FOCs are

$$(1) \ H_u = 0$$
$$(2) \ H_x = \rho\lambda(t) - \dot{\lambda}(t)$$

And, of course, don't forget the original constraint:

$$(3) \ \dot{x}(t) = g(x(t), u(t))$$

We also have

(4) "transversality condition",

which is satisfied in a lot of economic problems.

- Mathematicians derived this solution. We use it.
- These formulae will be given in exams.

3. The Neoclassical Growth Model Revisited

- Our problem:

$$\max_{\{C(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t)dt$$

s.t.

$$\dot{K}(t) = \underbrace{Y(t) - C(t)}_{=I(t)} - \delta K(t)$$
$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$$
$$A(t) = A(0)e^{g_A t}$$
$$L(t) = L(0)e^{g_L t}$$

- Eliminating $Y(t)$:

$$\max_{\{C_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t) dt$$

s.t.

$$\begin{aligned} \dot{K}(t) &= K(t)^{\alpha}(A(t)L(t))^{1-\alpha} - C(t) - \delta K(t) \\ A(t) &= A(0)e^{g_A t} \\ L(t) &= L(0)e^{g_L t} \end{aligned}$$

- We want to further eliminate $A(t)$ and $L(t)$.
- The textbook (Barro and Sala-i-Martin Appendix) has another way to solve the system. I prefer eliminating them.
- Define:

$$\begin{aligned} c(t) &= \frac{C(t)}{A(t)L(t)}, \\ k(t) &= \frac{K(t)}{A(t)L(t)}. \end{aligned}$$

- Then we have a simple form:

$$\begin{aligned} &\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{[c(t)A(t)]^{1-\sigma}}{1-\sigma} L(t) dt \\ &= \max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} \underbrace{[A(0)]^{1-\sigma}}_{\text{just constant}} e^{g_A(1-\sigma)t} \underbrace{L(0)}_{\text{just constant}} e^{g_L t} dt \\ &\Longrightarrow \max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho - g_A(1-\sigma) - g_L)t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt \\ &= \max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt, \end{aligned}$$

where $\eta \equiv \rho - g_A(1-\sigma) - g_L$ (later in the calibration we have to check $\eta > 0$).

- The constraint is

$$\begin{aligned} \frac{\dot{K}(t)}{A(t)L(t)} &= \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} - \frac{C(t)}{A(t)L(t)} - \delta \frac{K(t)}{A(t)L(t)} \\ &= k(t)^{\alpha} - c(t) - \delta k(t) \end{aligned}$$

We know

$$K(t) = k(t)A(t)L(t)$$

So

$$\dot{K}(t) = \dot{k}(t)A(t)L(t) + k(t)\dot{A}(t)L(t) + k(t)A(t)\dot{L}(t)$$
$$\implies \frac{\dot{K}(t)}{A(t)L(t)} = \dot{k}(t) + k(t)g_A + k(t)g_L$$

So the contraint becomes

$$\dot{k}(t) = k(t)^\alpha - c(t) - (\delta + g_A + g_L)k(t)$$

- Done!
- Applying Hamiltonian technique:
  - Control variable: $c(t)$
  - State variable: $k(t)$
  - h function: $\frac{[c(t)]^{1-\sigma}}{1-\sigma}$
  - g function: $k(t)^\alpha - c(t) - (\delta + g_A + g_L)k(t)$
- Hamiltonian function:

$$H = \frac{[c(t)]^{1-\sigma}}{1-\sigma} + \lambda(t)(k(t)^\alpha - c(t) - (\delta + g_A + g_L)k(t)).$$

FOCs are

$$H_c = 0$$
$$H_k = \eta\lambda(t) - \dot{\lambda}(t)$$

- So

$$c(t)^{-\sigma} - \lambda(t) = 0,$$
$$\text{and } \lambda(t)(\alpha k(t)^{\alpha-1} - (\delta + g_A + g_L)) = \eta\lambda(t) - \dot{\lambda}(t).$$
$$\text{So } \alpha k(t)^{\alpha-1} - (\delta + g_A + g_L) = \eta - \frac{\dot{\lambda}(t)}{\lambda(t)}$$

- We want to eliminate $\lambda(t)$.

- $\lambda(t) = c(t)^{-\sigma} \implies \log \lambda(t) = -\sigma \log c(t) \implies$ Take derivatives wrt t: $\frac{\dot{\lambda}(t)}{\lambda(t)} = -\sigma \frac{\dot{c}(t)}{c(t)}$. Hence,

$$\alpha k(t)^{\alpha-1} - (\delta + g_A + g_L) = \eta + \sigma \frac{\dot{c}(t)}{c(t)}$$

- Done! Let's recover our original notation:

$$\alpha \left(\frac{K(t)}{A(t)L(t)} \right)^{\alpha-1} - (\delta + g_A + g_L) = (\rho - g_A(1-\sigma) - g_L) + \sigma \overbrace{\left(\frac{\dot{C}(t)}{C(t)} - \underbrace{\frac{\dot{A}(t)}{A(t)}}_{g_A} - \underbrace{\frac{\dot{L}(t)}{L(t)}}_{g_L} \right)}^{\text{since } c(t)=C(t)/A(t)L(t)}$$

$$\alpha \left(\frac{K(t)}{A(t)L(t)} \right)^{\alpha-1} - (\delta + g_A + g_L) = (\rho - g_A - g_L) + \sigma\left(g_C - g_L\right)$$

$$\alpha \left(\frac{K(t)}{A(t)L(t)} \right)^{\alpha-1} - \delta = \rho + \sigma\left(g_C - g_L\right)$$

$$r - \delta = \rho + \sigma(g_C - g_L)$$

This is a continuous-time version of **Euler eq**.

- Constraints are basically the same as before. All other calibration results hold.
- But for Euler eq.,

$$8.2\% = \rho + \sigma 2.2\%$$

For example, if $\sigma = 1$ (log utility),

$$\rho = 8\% - 2.2\% \approx 6\%$$

- Well how is this related to our discrete-time version?

$$1 = \beta \left(\frac{C_{t+1}/L_{t+1}}{C_t/L_t} \right)^{-\sigma} (r_{t+1} + 1 - \delta)$$
$$1 = \beta(1.022)^{-\sigma} \times 1.08$$

If $\sigma = 1$,

$$\beta = \frac{1.022}{1.08} = 0.95.$$

But β^t (in a discrete-time version) was replaced by $e^{-\rho t}$ (in a continuous-time version). So β was replaced by $e^{-\rho}$. So $e^{-\rho} = e^{-0.06} = 0.94$. So they are almost the same.

EXERCISES

- Formula: For $\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t)) dt$ s.t. $\dot{x}(t) = g(x(t), u(t))$, $x(0)$ given,

$$H(x(t), u(t), \lambda(t)) = h(x(t), u(t)) + \lambda(t) g(x(t), u(t)),$$

and the first-order conditions are $H_u = 0$ and $H_x = \rho\lambda(t) - \dot{\lambda}(t)$.

1. Time is continuous. There is no uncertainty. Consider a closed, endowment economy with a representative consumer. The consumer solves at time 0

$$\max_{\{I(t),C(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \left[\log[H(t)] + C(t)\right] dt,$$

where $I(t)$ is the residential investment at period t, $H(t)$ is the stock of housing, $C(t)$ is the non-residential consumption, and $0 < \rho < 1$ is constant. (Notice that the logarithm is for $H(t)$ only.) The constraints are

$$Y(t) = I(t) + vH(t) \left(\frac{I(t)}{H(t)}\right)^2 + C(t),$$
$$\dot{H}(t) = I(t) - \delta H(t),$$

for all $t > 0$, where $\dot{H}(t) \equiv dh(t)/dt$. Here, $Y(t)$ is exogenous income (or endowment), and $v > 0$ and $\delta > 0$ are constant. Notice that the term $vH(t)\left(\frac{I(t)}{H(t)}\right)^2$ is the adjustment cost, reflecting the imperfect short run supply of resources to construction activities.

(a) Eliminate one of the two constraints and set up the Hamiltonian function. Introduce $\lambda(t)$ as a co-state variable (as in the "Hint" above).

Answer:

$$H = \log[H(t)] + Y(t) - I(t) - v\frac{[I(t)]^2}{H(t)} + \lambda(t)[I(t) - \delta H(t)]$$

(b) Obtain the first-order conditions. Write these conditions as

$$\lambda(t) = \boxed{\quad ? \quad}$$
$$\dot{\lambda}(t) = \lambda(t) \times \boxed{\quad ? \quad} + \boxed{\quad ? \quad}.$$

Answer:

$$H_I = -1 - 2v\frac{I(t)}{H(t)} + \lambda(t) = 0,$$
$$H_H = \frac{1}{H(t)} + v\frac{[I(t)]^2}{[H(t)]^2} - \delta\lambda(t) = \rho\lambda(t) - \dot{\lambda}(t).$$

Hence,

$$\lambda(t) = 1 + 2v\frac{I(t)}{H(t)},$$
$$\dot{\lambda}(t) = [\delta + \rho]\lambda(t) - \frac{1}{H(t)} - v\frac{[I(t)]^2}{[H(t)]^2}.$$

(c) What is your economic interpretation of $\lambda(t)$? Write in plain English.

Answer: *It is the price of housing, or the marginal cost of housing. The first equation, $\lambda(t) = 1 + 2v\frac{I(t)}{H(t)}$, implies that $\lambda(t)$ is the marginal cost of housing. That is, to build an additional unit of housing, one needs to invest one unit of physical good, but an additional cost is $2v\frac{I(t)}{H(t)}$, which is the first-order derivative of adjustment cost.*

(d) Consider a "steady state" in which all variables (including, of course, $Y(t)$ and $\lambda(t)$) are constant. This is like a balanced growth path in which all growth rates are zero. Recall that $Y(t)$ is exogenously given, and hence, the consumer knows its steady-state level in advance. Solve for the steady-state levels of $I(t)$ and $H(t)$.

Answer: *The constraints and the first-order conditions become*

$$Y = I + v\frac{I^2}{H} + C,$$
$$0 = I - \delta H,$$
$$\lambda = 1 + 2v\frac{I}{H},$$
$$0 = [\delta + \rho]\lambda - \frac{1}{H} - v\frac{I^2}{H^2}.$$

Eliminating λ,

$$Y = I + v\frac{I^2}{H} + C,$$
$$\frac{I}{H} = \delta,$$
$$0 = [\delta + \rho]\left[1 + 2v\frac{I}{H}\right] - \frac{1}{H} - v\frac{I^2}{H^2}.$$

This is an equation system for three unknowns, C, I and H. Eliminating I using the second equation, the third equation becomes

$$0 = [\delta + \rho]\,[1 + 2v\delta] - \frac{1}{H} - v\delta^2.$$

Hence,

$$\frac{1}{H} = [\delta + \rho]\,[1 + 2v\delta] - v\delta^2,$$
$$\text{So } H = \frac{1}{[\delta + \rho]\,[1 + 2v\delta] - v\delta^2}$$

and

$$I = \frac{\delta}{[\delta + \rho]\,[1 + 2v\delta] - v\delta^2}.$$

International Knowledge Diffusion and Growth

References

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- [O] Luckstead, J., S. M. Choi, S. Devadoss, and R. C. Mittelhammer (2012), "A Decomposition of China's Productivity through Calibration of the Neoclassical Growth Model," unpublished manuscript.
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- **Data**: "International Knowledge Diffusions" in EconS 427
  - The **income-doubling time** (from $2,000 to $4,000) has decreased.
    - Possible interpretation: There is an international knowledge diffusion.
  - The economies open to **international trade** converge in per-capita GDP.
    - Possible interpretation: The international knowledge diffusion is related to international trade.
  - The share of **agriculture** out of GDP decreases as the per-capita GDP grows.
    - Possible interpretation: The productivity of non-agriculture grows faster. Perhaps this is because the "land" is fixed. Or perhaps there is more "learning by doing" in non-agriculture.
  - The **manufacturing export** share out of GDP tends to grow as the per-capita GDP grows.

* Possible interpretation: Perhaps it is important to design a policy so that the economy focuses on manufacturing exports. (i) Higher manufacturing production provides more learning by doing. (ii) You compete to sell manufacturing products in global market. You learn to improve your productivity. You learn from other economies, too.

- **Need**: A theoretical, quantifable model that is consistent with all of the above.
- Surprisingly enough, we do not have such a model yet. However, let's try to approach with an exiting model.

1. A Mechanical Model of Catch-Up

- Simple. Productivity only. There is no K accumulation, etc.
- **Leading Economy** (Economy 1):

$$\text{Output per worker: } y_1(t) = \underbrace{A_1(t)}_{\text{Productivity}}.$$

$$\text{Productivity growth: } \dot{A}_1(t) = \underbrace{g}_{\text{const growth}} A_1(t).$$

$$\text{So } \frac{\dot{A}_1(t)}{A_1(t)} = g.$$

- **Catch-up Economy** (Economy 2, with lower $A_2(t)$):

$$\text{Output per worker: } y_2(t) = A_2(t).$$

$$\text{Productivity growth: } \dot{A}_2(t) = g\left[A_2(t)\right]^{1-\theta}\left[A_1(t)\right]^{\theta},\ 0 < \theta < 1$$

$$\text{So } \frac{\dot{A}_2(t)}{A_2(t)} = g\left[\frac{A_1(t)}{A_2(t)}\right]^{\theta}.$$

- (1) Knowledge diffusion from 1 to 2. Hence, $A_1(t)$ enters.
- (2) As $A_2(t) \longrightarrow A_1(t)$, 2's growth rate converges to 1's. This is why we use CRS Cobb-Douglas.
- θ is the "speed" of convergence. If 0, no convergence. If high, fast.
- **Calibration**:

- g is the growth rate of per-capita output for a high-income economy. Use the average of all high-income economies. Or use the U.S. or the U.K.

- Then $g = 2\%$ or so.

- For your Economy 2, select the best θ (e.g., by regression). That is, from

$$\frac{\dot{y}_2(t)}{y_2(t)} = g\left[\frac{y_1(t)}{y_2(t)}\right]^{\theta},$$

we have

$$\log\left(\frac{\dot{y}_2(t)}{y_2(t)}\right) = \log g + \theta \log\left(\frac{y_1(t)}{y_2(t)}\right).$$

2. Lucas (2009): Agriculture and Manufacturing

- Assume: GDP = Agricultural production + Manufacturing production

- Motivation: Ag employment share declines as per-capita GDP grows. The cities provide an "engine" of growth.

- Employment Share of Agriculture in 1960:

  Hong Kong: 8%

  South Korea: 66%

  Indonesia: 75%

  Thailand: 84%

  (Korea vs. Indonesia is not a big difference?)

- Two sectors, "city" and "farm".

- Labor is allocated to "city" (fraction $1 - x$) and "farm" (fraction x).

- **City production**: $y_{2,city}(t) = [1 - x_2(t)]\, h_2(t)$

  (h_2: human capital per worker, or worker productivity)

- **Farm production**: $y_{2,farm}(t) = A_2 \underbrace{[h_2(t)]^{\xi}}_{\text{spillover}} [\underbrace{x_2(t)}_{\text{raw labor}}]^{\alpha} \underbrace{(Land)^{1-\alpha}}_{=1 \text{ (const)}}.$

  ξ =psi: "Spillover" from "city" to "farm".

  The source of the farm's productivity growth.

  $0 < \alpha < 1$: The land is fixed.

- **Problem**: Maximize the GDP, $y_2(t) = y_{2,city}(t) + y_{2,farm}(t)$.

$$\max_{x_2(t)} (1 - x_2(t))h_2(t) + A_2 h_2(t)^{\xi} x_2(t)^{\alpha}.$$

FOC:

$$h_2(t) = A_2 h_2(t)^{\xi} \alpha x_2(t)^{\alpha-1}.$$

$$\text{So } x_2(t) = \left(\frac{h_2(t)^{1-\xi}}{\alpha A_2}\right)^{1/(\alpha-1)}$$

- What we have: Optimal allocation of workers between Ag and Manufacturing when human capital per worker is given.

- How does the human capital accumulate?

- **Assumption**:

Original Model	This Model
$\dot{A}_2(t) = g\,[A_1(t)]^{1-\theta}\,[A_2(t)]^{\theta}$	$\dot{h}_2(t) = \mu \Big[\underbrace{1 - x_2(t)}_{\text{fraction of workers in city}}\Big]^{\zeta} [h_2(t)]^{1-\theta}\,[h_1(t)]^{\theta}$
	$= \mu \left[1 - \left(\frac{h_2(t)^{1-\xi}}{\alpha A_2}\right)^{1/(\alpha-1)}\right]^{\zeta} h_2(t)^{1-\theta} h_1(t)^{\theta}$

ζ =zeta

- For Economy 1 (leader),
  - Productions same. Optimal allocation of workers same.
  - Human capital accumulation: $\dot{h}_1(t) = \mu \left[1 - \left(\frac{h_1(t)^{1-\xi}}{\alpha A_1}\right)^{1/(\alpha-1)}\right]^{\zeta} h_1(t)$.

- **Calibration**: Assume $\alpha = 0.6$, $\mu = 2\%$, $\theta = 0.67$, $\xi = 0.75$ and $\zeta = 1$.

  (These numbers seem to have been obtained with trials and errors. Better to justify with data.)

- Given: For both economies, per-capita GDP and its fraction of workers in farm, in initial year: $y_2(1960)$ and $x_2(1960)$

- We can solve for $h_2(1960)$ and A_2 in the following equation system:

(1) GDP per capita: $\underbrace{y_2(1960)}_{\text{data}} = (1 - \underbrace{x_2(1960)}_{\text{data}})h_2(1960) + A_2\,(h_2(1960))^{\xi}\,(\underbrace{x_2(1960)}_{\text{data}})^{\alpha}$

$$\text{(2) FOC: } \underbrace{x_2(1960)}_{\text{data}} = \left(\frac{h_2(1960)^{1-\xi}}{\alpha A_2}\right)^{1/(\alpha-1)}$$

- Obtain $h_1(1960)$ and A_1 with this method, too.
- Then the growth of $h_2(t)$ is now obtained:

$$\frac{\dot{h}_2(t)}{h_2(t)} = \underbrace{\mu}_{0.02}\underbrace{\left[1 - \left(\frac{h_2(1960)^{1-\xi}}{\alpha A_2}\right)^{1/(\alpha-1)}\right]}_{\text{known}}^{\overbrace{\zeta}^{1}} \left[\underbrace{\frac{h_1(t)}{h_2(t)}}_{\text{known}}\right]^{\overbrace{\theta}^{0.67}}$$

- This gives $h_2(1961)$.
- Then, $x_2(1961)$ and $y_2(1961)$ follow from (1) and (2).
- The counterparts for Economy 1 can be obtained similarly.
- Continue forever.
- **Results**:
- FIGURE: Per-capita GDp vs. Employment share of Ag (Lucas (2009), Figure 14.)
- FIGURE: Hong Kong's Per-capita GDP Growth, Observed vs. Predicted (Lucas (2009), Figure 18.)
- FIGURE: South Korea's Per-capita GDP Growth, Observed vs. Predicted (Lucas (2009), Figure 19.)
- FIGURE: Indonesia's Per-capita GDP Growth, Observed vs. Predicted (Lucas (2009), Figure 20.)
- FIGURE: Thailand's Per-capita GDP Growth, Observed vs. Predicted (Lucas (2009), Figure 21.)
- **Conclusion**: Initial "mix" of ag vs. manufacturing (or farm vs city) matters.

- What is missing in this model?
  - What determines the speed of convergence, θ?
  - And how does international trade affect this speed? What is the role of manufacturing exports, especially?
- Need: An introduction of international trade.

EXERCISES

1. From what year do you expect China's GDP (not GDP per capita) to exceed the U.S. GDP?

   Answer: *There are several qualified models. The model should at least have a "catch-up growth" feature. Here is a possible model that I would use. Good features: (i) There is a catch-up. (ii) Physical capital accumulation is controlled for when we estimate China's productivities. Features to be improved: No education. No agricultural sector, etc. (Perhaps not big deals, though.)*

$$
\begin{aligned}
Y_{US}(t) &= K_{US}(t)^{\alpha}[A_{US}(t)L_{US}(t)]^{1-\alpha},\\
\dot{K}_{US}(t) &= I_{US}(t) + (1-\delta)K_{US}(t),\\
\dot{A}_{US}(t) &= \mu A_{US}(t),\\
\dot{L}_{US}(t) &= \lambda_{US}L_{US}(t).
\end{aligned}
$$

   For China,

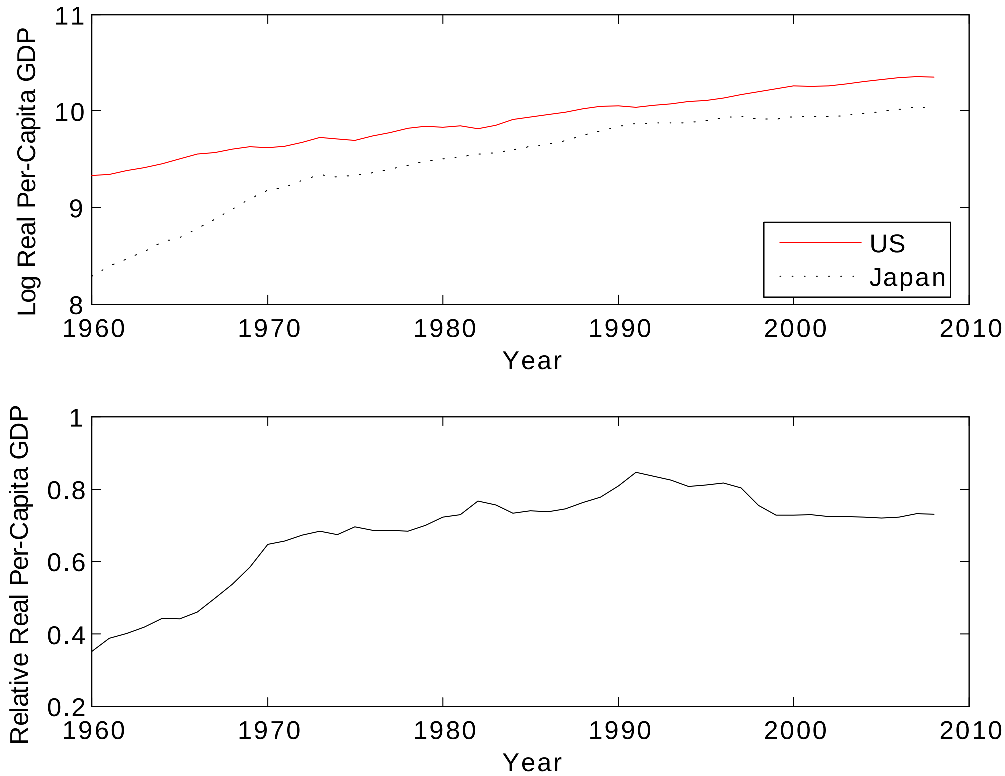
$$
\begin{aligned}
Y_{CHN}(t) &= K_{CHN}(t)^{\alpha}[A_{CHN}(t)L_{CHN}(t)]^{1-\alpha},\\
\dot{K}_{CHN}(t) &= I_{CHN}(t) + (1-\delta)K_{CHN}(t),\\
\dot{A}_{CHN}(t) &= \mu\left[A_{CHN}(t)\right]^{1-\theta}\left[A_{US}(t)\right]^{\theta},\\
\dot{L}_{CHN}(t) &= \lambda_{CHN}L_{CHN}(t).
\end{aligned}
$$

   Here, θ is a catch-up coefficient.

   For calibration, we may assume $\alpha = 1/3$ (which is standard) or compute the labor income share from the data. $L(t)$ is population and the population growth, λ, can be easily obtained from data. (You may want to assume a more sophisticated law of motion of population.) We have $\frac{\dot{K}_{US}(t)}{K_{US}(t)} = \frac{I_{US}(t)}{Y_{US}(t)}\frac{Y_{US}(t)}{K_{US}(t)} + (1-\delta)$, and we can show that $\frac{\dot{K}_{US}(t)}{K_{US}(t)} = \frac{\dot{Y}_{US}(t)}{Y_{US}(t)}$ (by the assumption that the interest rate is constant), and we can obtain $\frac{\dot{Y}_{US}(t)}{Y_{US}(t)}$ (GDP growth rate) and $\frac{I_{US}(t)}{Y_{US}(t)}$ (investment rate) from the data. Hence, we can solve for $\frac{Y_{US}(t)}{K_{US}(t)}$ (an inverse of K/Y), either from an assumption $\delta = 0.06$ (standard) or using the U.S. "consumption of fixed capital" data. This gives us $K_{US}(t)$ for all t. Since we have $Y_{US}(t)$, $K_{US}(t)$ and $L_{US}(t)$, the first equation gives $A_{US}(t)$ for all *t. Then μ is straightforward. We can apply similar steps for China and we can obtain θ, using, for example, the OLS, from the data from 1985 or so (when China opened). The model is now quantified.*

Now we simulate. This quantified model will provide the path of per-capita GDP, $Y(t)/L(t)$, because we know how $K(t)$, $h(t)$ and $L(t)$ in the two economies grow over time and we have initial values. The U.S. will show a constant growth rate. China's growth rate of per-capita GDP will eventually converge to the U.S. level, but the growth rate will also decline, also converging to the U.S. level. Using this and $L(t)$ (exogenously growing), we can compute $Y(t)$ for both economies.Then we know in what year $Y_{CHN}(t)$ will exceed $Y_{US}(t)$.

2. The top figure shows the logarithms of real per-capita GDPs of the United States and Japan in 1960-2008. The bottom figure shows their ratio (i.e., the real per-capita GDP of Japan relative to the United States).



(a) Japan's real per-capita GDP features catch-up growth. Suppose you want to model the evolutions of real per-capita GDPs, $\{y_{USA,t}\}$ and $\{y_{JPN,t}\}$, as follows:

$$\frac{\dot{y}_{USA,t}}{y_{USA,t}} = g,$$
$$\frac{\dot{y}_{JPN,t}}{y_{JPN,t}} = \boxed{\quad ? \quad},$$

where g is a constant and $\dot{y}_{USA,t} \equiv dy_{USA,t}/dt$. Fill the blank, using $y_{USA,t}$, $y_{JPN,t}$, g and one or more parameters you want to introduce. Notice that the catch-up appears to have completed around 1985.

Answer: *A simple way to model it is to have* $\frac{\dot{y}_{JPN}}{y_{JPN}} = g\left(\frac{y_{USA}}{y_{JPN}}\right)^{\theta}$ *as in class. Then, as* y_{JPN} *reaches* y_{USA}*, the growth rate of Japan converges to the growth rate of the United States. [This model is good, but not creative for a PhD student.]*

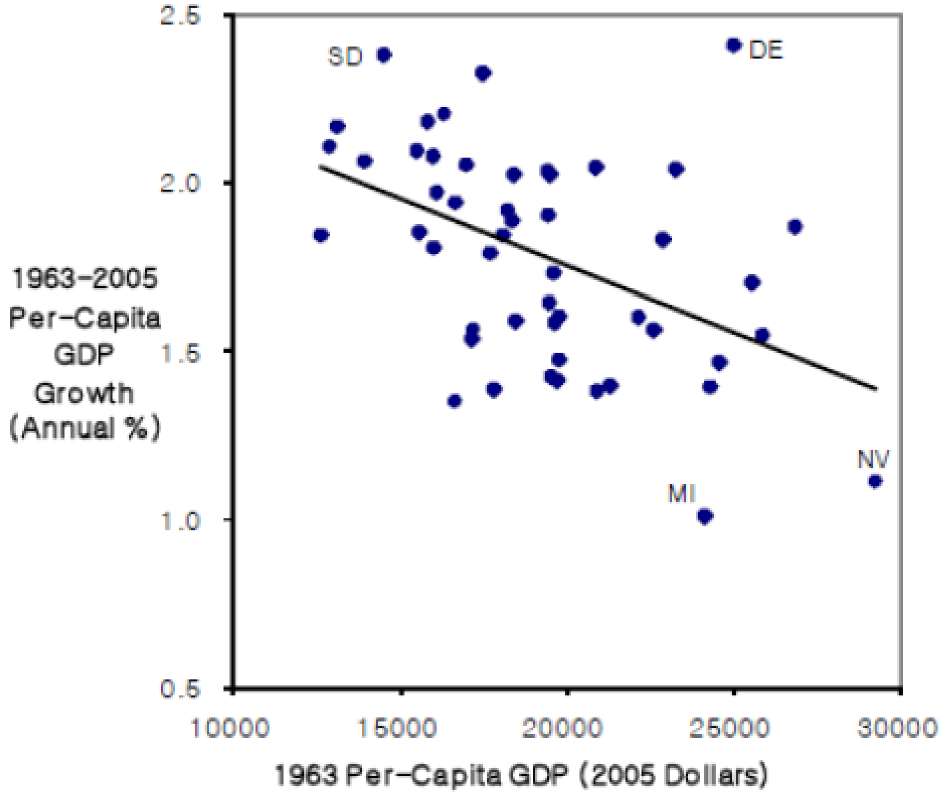
However, when the catch-up was completed around 1985, $\frac{y_{USA}}{y_{JPN}}$ *was not one. It is about 0.8 or so. Hence,* $\frac{\dot{y}_{JPN}}{y_{JPN}} = g\left(\mu\frac{y_{USA}}{y_{JPN}}\right)^{\theta}$ *(where* μ *is 0.8 or so) will be a better model that fits the data.*

(b) Discuss how you will calibrate g and the parameter(s) you introduced, using the data on real per-capita GDPs of the two economies.

Answer: *g is the average of U.S. growth rates. It will be around 2%. For* $\frac{\dot{y}_{JPN}}{y_{JPN}} = g\left(\frac{y_{USA}}{y_{JPN}}\right)^{\theta}$*, take logs to have* $\log(\frac{\dot{y}_{JPN}}{y_{JPN}}) = \log g + \theta \log\left(\frac{y_{USA}}{y_{JPN}}\right)$*. We know* $\log g$*. Then,* θ *can be estimated using average levels of* $\frac{\dot{y}_{JPN}}{y_{JPN}}$ *and* $\frac{y_{USA}}{y_{JPN}}$ *in the sample. Or we can simply run the OLS (but a problem is that* $\frac{\dot{y}_{JPN}}{y_{JPN}}$ *is often negative, so its logarithm does not exist.)*

For $\frac{\dot{y}_{JPN}}{y_{JPN}} = g\left(\mu\frac{y_{USA}}{y_{JPN}}\right)^{\theta}$*, we know* g*, and we know* μ *is 0.8 or so. (For example, you can use the 1985-2008 average of relative GDPs.) Then again, take logs and estimate* θ *as above.*

3. Treat all U.S. States as separate economies. The following figure shows that in 1963-2005, per-capita GDPs of the States had been converging.



To explain this convergence, you consider the following hypotheses: (i) Physical capital moved from the States with lower interest rates to the States with higher interest rates. (ii) The States with lower productivity benefits from the knowldge spillover from the States with higher productivity.

(a) To address hypothesis (i), you assume that State i (for all i) follows a production function at period t:

$$Y_{it} = (K_{it})^{\alpha}(A_t L_{it})^{1-\alpha},$$

for $0 < \alpha < 1$, where Y_{it} is the real GDP, K_{it} is the physical capital stock, A_t is the productivity, and L_{it} is the population. Notice that A_t is common across all States. Assume that only physical capital (and not labor) can move across the States. Discuss how the movement of physical capital potentially explains the convergence of per-capita GDP. In particular,

i. If there are two States with higher and lower levels of per-capita physical capital stocks (K_{it}/L_{it}), to what direction does the physical capital move?

ii. When this happens, will the per-capita income (Y_{it}/L_{it}) converge over time?

Answer: *The interest rate is*

$$\begin{aligned} r_{it} &= \alpha (K_{it})^{\alpha-1}(A_t L_{it})^{1-\alpha} \\ &= \alpha (A_t)^{1-\alpha}\left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} \end{aligned}$$

A State with higher K/L has lower r. Hence, physical capital move from such States to the States with lower K/L. This will make per-capita income converge because

$$\frac{Y_{it}}{L_{it}} = \left(\frac{K_{it}}{L_{it}}\right)^{\alpha}(A_t)^{1-\alpha},$$

i.e., per-capita income is positively related to per-capita physical capital stock.

(b) To address hypothesis (ii), you assume that State i follows a production function at period t:

$$Y_{it} = A_{it}L_{it},$$

where the productivity A_{it} now differs across the States. Furthermore, the evolution of A_{it} follows

$$\dot{A}_{it} = \underline{\hspace{3cm}}.$$

i. Fill the blank on your own way. Clearly justify your equation.

ii. Briefly discuss how you will estimate (or calibrate) your equation in (i) using the data observations that are used to draw the above figure.

Answer: *There can be several possible approaches. For example, one can specify the evolution of productivity as*

$$\dot{A}_{it} = g_A (A_{it})^{1-\theta}(A_{1t})^{\theta},$$

where A_{1t} is the productivity of the "leader", which is NV (Nevada) in the above figure. A parameter θ captures the knowledge spillover. Given g_A, we write the equation as

$$\frac{\dot{A}_{it}}{A_{it}} = g_A\left(\frac{A_{1t}}{A_{it}}\right)^{\theta}.$$

Taking logs,

$$\log\left(\frac{\dot{A}_{it}}{A_{it}}\right) = \log g_A + \theta \log\left(\frac{A_{1t}}{A_{it}}\right).$$

Since $A_{it} = Y_{it}/L_{it}$ (per-capita GDP), the data in the above figure is enough to run this regression in the cross section. (Of course, we have only about 50 observations.)

4. Suppose that a long-term investor is interested in emerging markets. He/she is asking for your predictions on the long-run economic growth (up to 2050 or so) in various emerging economies. You have decided to follow the approach in Lucas, Robert E., Jr. (2009), "Trade and the Diffusion of the Industrial Revolution," American Economic Journal: Macroeconomics, 1(1), 1-25.

   (a) Duplicate one of the four figures: Figure 18 (Hong Kong), Figure 19 (South Korea), Figure 20 (Indonesia), and Figure 21 (Thailand). Use the (calibrated) parameter values reported in Lucas (2009). Also use the "initial values" reported in each figure. As in Lucas (2009), use Angus Maddison's data for per-capita GDP. (Just show two lines, predicted and actual growth rates. You don't have to show the "trend" of actual growth rate.)

   (b) Now provide another figure that shows the predicted growth rate of per-capita output from the last year in data to 2050. Use the most recent observation on agricultural employment share from the World Development Indicators.

   (c) Now pick up any other economy in the world, other than the one you chose in (a). Call it Economy A. Your client is asking for a prediction on Economy A's economic growth. Provide a figure that shows the predicted growth rates of per-capita output from the last year in data to 2050. Use the most recent observation on agricultural employment share from the World Development Indicators. Assume Economy A is "open" in the last year in data, so that you can apply Lucas's (2009) model.

International Trade and Growth

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1. Motivation

- "Trade $\longrightarrow$ Growth" not fully explored.
- **Discussions**:
  - Some argue that the catch-up starts after the "opening".
  - However, opening up a domestic market may force domestic manufacturing firms to fail. (Young, 1991.)
  - But **exports** may be important. Firms should compete in an international market (i.e., play in a big league) to learn technologies, know-how, etc.
  - **Example**: Korea became the 4th largest producer of automobiles. 6th largest exporter.
    * However, the domestic market had been heavily protected for decades. Japan's Toyota Lexus entered only in 1999-2000. Japan's Honda entered in 2004. Even now, there are high tariffs.
    * Korea's productivity growth in automobiles was not from opening the domestic market, but from exporting to an international market.

- In short, we need a model that more clearly connects international trade (and especially manufacturing exports) to growth.
- How do we connect manufacturing exports to productivity growth? A candidate is **learning by doing** (LBD).
- Choi, Kim and Ma (2011), in comparison w/ Lucas (2009):

Lucas (2009)	Choi, Kim and Ma (2011)
International knowledge diffusion	**Learning by doing**
City is a place for human capital accumulation. **Farm** enjoys a knowledge diffusion from City.	**Manufacturing** is a sector for productivity growth by LBD. Same.

2. Learning by Doing in a Closed Economy

- Lecture notes not completed – See Choi, Kim and Ma (2012).
- **Digression: Some properties (tricks?) of Cobb-Douglas**
  - C-D utility function $u(t) = C_1(t)^\alpha C_2(t)^\beta$.
  - Implication: $\frac{\alpha}{\alpha+\beta}$ is a share of income to spend on good 1. And $\frac{\beta}{\alpha+\beta}$ is a share of income to spend on good 2.
  - To see this,
$$\max_{C_1,\, C_2} C_1^\alpha C_2^\beta \quad \text{s.t. } p_1 C_1 + p_2 C_2 = M.$$
  - $\Longrightarrow \max_{C_1} C_1^\alpha \left(\frac{M - p_1 C_1}{p_2}\right)^\beta$.
  - FOC: $\alpha C_1^{\alpha-1}\left(\frac{M-p_1C_1}{p_2}\right)^\beta + C_1^\alpha \beta \left(\frac{M-p_1C_1}{p_2}\right)^{\beta-1}\left(-\frac{p_1}{p_2}\right) = 0$.
  - $\Longrightarrow \alpha C_1^{\alpha-1}\left(\frac{M-p_1C_1}{p_2}\right)^\beta = C_1^\alpha \beta \left(\frac{M-p_1C_1}{p_2}\right)^{\beta-1}\left(\frac{p_1}{p_2}\right)$.
  - $\Longrightarrow \alpha \left(\frac{M-p_1C_1}{p_2}\right) = C_1 \beta \left(\frac{p_1}{p_2}\right)$.
  - $\Longrightarrow \alpha C_2 = C_1 \beta \left(\frac{p_1}{p_2}\right)$.
  - $\Longrightarrow \underbrace{\frac{\alpha}{\beta} C_2 p_2}_{\text{\$ spent on good 2}} = \underbrace{C_1 p_1}_{\text{\$ spent on good 1}}$
  - $\Longrightarrow$ (\$ spent on good 1) is (\$ spent on good 2) multifiled by $\frac{\alpha}{\beta}$.
  - \$$M$ are allocated as follows. A fraction $\frac{\alpha}{\alpha+\beta}$ goes to good 1. A fraction $\frac{\beta}{\alpha+\beta}$ goes to good 2.

EXERCISES

1. There is a mass one of identical workers. Two types of goods, N (non-agriculture) and A (agriculture), exist. There are no storage technologies. Each worker owns one non-agricultural firm producing N goods, and one farm producing A goods. At each period t, the worker allocates a fraction $0 \leq l_t \leq 1$ of his time into non-agricultural firm and $1 - l_t$ into farm. Each non-agricultural firm's production function is

$$Y_t^N = N_t l_t,$$

where Y_t^N is the output in units of N goods, and N_t is the economy-wide productivity in N sector. Here, N_t evolves externally according to the economy-wide production, following

$$N_{t+1} - N_t = \mu N_t \bar{l}_t,$$

for $\mu > 0$, where $\bar{l}_t$ is an economy-wide level of l_t (so in equilibrium, $\bar{l}_t = l_t$). Since an individual firm is small, its production decision does not affect the evolution of economy-wide productivity.

Each farm's production function is

$$Y_t^A = A\left(1 - l_t\right)^{\phi},$$

where Y_t^A is the output in units of A goods. Here, A is a constant which captures cross-country differences in fertilities, etc. A parameter ϕ determines the curvature of A production, where a condition $0 < \phi < 1$ is imposed to reflect decreasing marginal products due to fixed amount of lands, etc.

This is a small open economy, taking (an evolution of) "world prices," $\{p_t^N\}$ and $\{p_t^A\}$, exogenously given. There are no tariffs or other barriers to trade.

   (a) Since the individual decision of l_t in period t does not affect the variables in all future periods, $t+1$, $t+2$, ..., the decision depends on the revenues only of the current period. Provided that each worker maximizes the aggregate revenue from N Sector and A Sector, he will solve

$$\max_{l_t} p_t^N N_t l_t + p_t^A A(1 - l_t)^{\phi},$$

   given N_t, p_t^N and p_t^A, respectively. Obtain the first-order condition. Write the first-order condition explicitly for l_t:

$$l_t = \boxed{\quad ? \quad}.$$

Answer: *The first-order condition is*

$$p_t^N N_t = \phi p_t^A A(1-l_t)^{\phi-1},$$

implying that the values of marginal products of labor in two sectors are equated. Hence,

$$(1-l_t)^{\phi-1} = \frac{1}{\phi}\frac{p_t^N}{p_t^A}\frac{N_t}{A}.$$

$$\text{So } 1-l_t = \left(\frac{1}{\phi}\frac{p_t^N}{p_t^A}\frac{N_t}{A}\right)^{\frac{1}{\phi-1}}.$$

$$\text{So } l_t = 1-\left(\frac{1}{\phi}\frac{p_t^N}{p_t^A}\frac{N_t}{A}\right)^{\frac{1}{\phi-1}}.$$

(b) Based on your answer to (a), determine how an economy's productivity growth (i.e., $\frac{N_{t+1}-N_t}{N_t}$) is affected by world prices. Provide your economic interpretation about why it happens.

Answer: *From (a),*

$$\frac{N_{t+1}-N_t}{N_t} = \mu l_t$$
$$= \mu\left[1-\left(\frac{1}{\phi}\frac{p_t^N}{p_t^A}\frac{N_t}{A}\right)^{\frac{1}{\phi-1}}\right].$$

This implies that as the world's relative price, $\frac{p_t^N}{p_t^A}$, is higher, productivity growth is also higher. (You can see this by taking a first derivative with respect to $\frac{p_t^N}{p_t^A}$. Or, it is simply clear from the fact that $\frac{1}{\phi-1} < 0$.)

Interpretation: If $\frac{p_t^N}{p_t^A}$ is higher, it will be more profitable to engage in N industry. But since learning by doing is in N industry only, the productivity will grow faster.

(c) Suppose that there are many small open economies. Based on your answer to (a), discuss which economy's productivity growth (i.e., $\frac{N_{t+1}-N_t}{N_t}$) tends to be higher, given world prices. Provide your economic interpretation about why it happens.

Answer: *An economy with a higher level of relative productivity, $\frac{N_t}{A}$, faces higher productivity growth.*

Interpretation: If $\frac{N_t}{A}$ is higher, this economy has a comparative advantage in N sector. Hence, it will focus more on N sector. But since learning by doing is in N industry only, the productivity will grow faster.

R&D, Technological Advance, and Growth

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1. A School vs. H School

- $Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$
- So what is $A(t)$?
  1. Technological advances
  2. Education
  3. Others
- We separate 1 and 2: $Y(t) = K(t)^{\alpha}(A(t)h(t)L(t))^{1-\alpha}$.
-

A School	**H School**
Once technology is found, it is non-rival and (almost) free. One genius is important. Focus on R&D, patent system, ...	To learn the knowledge, everyone spends time/resources. Average worker is important. Focus on schooling, OJT, ...

- They are not necessarily mutually exclusive. It is often difficult to distinguish between the two.

- So what is the view of the "A school"? ("H school" in the next chapter.)

2. Example: Price Decline in Capital Goods

- Greenwood, Hercowitz and Krusell (1997).
- We have better computers, better telecommunication and transportation, robotization of assembly lines, ...
- Machine (of the same quality) becomes cheaper!
- Look at **price decline** of **capital goods** (labeled as "investment goods" in NIPA) compared to **physical goods**.
- Data: NIPA Table 1.1.4
-

	1960	2000	2005
General Price Index	21.044	100.00	113.039
Price index for capital good (s =20%)	29.619	100.00	111.381
Price index for consumptiong ood (80%)	18.900*	100.00	113.454

  * Obtained by solving $21.044 = 29.619 \times 0.2 + X \times 0.8$.

- Growth of price index for capital good: $\frac{\log(111.381)-\log(29.619)}{45}$ =2.9%
- Growth of price index for consumption good: $\frac{\log(113.454)-\log(18.900)}{45}$ =4.0%
- Hence, capital goods become cheaper compared to physical goods!
- Model:

$$\max_{\{C(t),I(t)\}_{t=0}^{\infty}} \int_0^\infty e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t)dt \text{ (same)}$$

  s.t.

$$\begin{aligned} Y(t) &= C(t) + I(t) \text{ (same)} \\ \dot{K}(t) &= \underbrace{q(t)}_{\text{What is this?}} I(t) - \delta K(t) \text{ (different)} \end{aligned}$$

- For example,

$$\begin{aligned} \underbrace{Y(t)}_{100 \text{ apples}} &= \underbrace{C(t)}_{80 \text{ apples}} + \underbrace{I(t)}_{20 \text{ apples}} \text{ (same)} \\ \dot{K}(t) &= \underbrace{\underbrace{q(t)}_{=1/2} I(t)}_{20 \text{ applies transformed to 10 trees}} - \delta K(t) \text{ (different!)} \end{aligned}$$

- So physical goods (apples) and capital goods (trees) are different.
- $I(t)$ units of physical goods (apples) are transformed to $q(t)I(t)$ units of capital goods (trees).

  $\Longrightarrow q(t)$ is the **productivity of capital goods production**

  $1/q(t)$ is the **price of a capital good** in units of physical goods
- We also have

$$\begin{aligned} Y(t) &= K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \text{ (same)} \\ A(t) &= A(0)e^{g_A t} \text{ (same)} \\ L(t) &= L(0)e^{g_L t} \text{ (same)} \\ q(t) &= q(0)e^{g_q t} \text{ (new!)} \end{aligned}$$

- Eliminating $I(t)$ and $Y(t)$:

$$\max_{\{C(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t) dt$$

  subject to

$$\begin{aligned} \dot{K}(t) &= q(t)[\underbrace{K(t)^{\alpha}(A(t)L(t))^{1-\alpha} - C(t)}_{=I(t)}] - \delta K(t) \\ A(t) &= A(0)e^{g_A t} \\ L(t) &= L(0)e^{g_L t} \\ q(t) &= q(0)e^{g_q t} \end{aligned}$$

- Need: Eliminate $A(t)$, $L(t)$ and $q(t)$.
- Define:

$$\begin{aligned} c(t) &= \frac{C(t)}{q(t)^{\alpha/(1-\alpha)} A(t)L(t)}, \\ k(t) &= \frac{K(t)}{q(t)^{1/(1-\alpha)} A(t)L(t)}. \end{aligned}$$

- Then we have a simple form:

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{[c(t) q(t)^{\alpha/(1-\alpha)} A(t)]^{1-\sigma}}{1-\sigma} L(t) dt$$

$$= \max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} \underbrace{[q(0)]^{(1-\sigma)\alpha/(1-\alpha)}}_{\text{just constant}} e^{g_q[(1-\sigma)\alpha/(1-\alpha)]t} \underbrace{[A(0)]^{1-\sigma}}_{\text{just constant}} e^{g_A(1-\sigma)t} \underbrace{L(0)}_{\text{just constant}} e^{g_L t} dt$$

$$\implies \max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho - g_q[(1-\sigma)\alpha/(1-\alpha)] - g_A(1-\sigma) - g_L)t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt$$

$$= \max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt$$

where

$$\eta \equiv \rho - g_q[(1-\sigma)\alpha/(1-\alpha)] - g_A(1-\sigma) - g_L$$

- Subject to

$$\begin{aligned}
\frac{\dot{K}(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} &= \frac{q(t)[K(t)^{\alpha}(A(t)L(t))^{1-\alpha} - C(t)]}{q(t)^{1/(1-\alpha)} A(t) L(t)} - \delta \frac{K(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} \\
&= \frac{K(t)^{\alpha}(A(t)L(t))^{1-\alpha}}{q(t)^{\alpha/(1-\alpha)} A(t) L(t)} - \frac{C(t)}{q(t)^{\alpha/(1-\alpha)} A(t) L(t)} - \delta \frac{K(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} \\
&= \frac{K(t)^{\alpha}}{q(t)^{\alpha/(1-\alpha)} [A(t) L(t)]^{\alpha}} - \frac{C(t)}{q(t)^{\alpha/(1-\alpha)} A(t) L(t)} - \delta \frac{K(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} \\
&= \left(\frac{K(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} \right)^{\alpha} - \frac{C(t)}{q(t)^{\alpha/(1-\alpha)} A(t) L(t)} - \delta \frac{K(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} \\
&= k(t)^{\alpha} - c(t) - \delta k(t)
\end{aligned}$$

- How to deal with LHS? Since $K(t) = k(t) q(t)^{1/(1-\alpha)} A(t) L(t)$,

$$\begin{aligned}
\dot{K}(t) &= \dot{k}(t) q(t)^{1/(1-\alpha)} A(t) L(t) + k(t) \frac{1}{1-\alpha} q(t)^{1/(1-\alpha)-1} \dot{q}(t) A(t) L(t) \\
&\quad + k(t) q(t)^{1/(1-\alpha)} \dot{A}(t) L(t) + k(t) q(t)^{1/(1-\alpha)} A(t) \dot{L}(t)
\end{aligned}$$

So

$$\frac{\dot{K}(t)}{q(t)^{1/(1-\alpha)} A(t) L(t)} = \dot{k}(t) + \frac{1}{1-\alpha} k(t) \underbrace{\frac{\dot{q}(t)}{q(t)}}_{=g_q} + k(t) \underbrace{\frac{\dot{A}(t)}{A(t)}}_{=g_A} + k(t) \underbrace{\frac{\dot{L}(t)}{L(t)}}_{=g_L}$$

- So the constraint is

$$\dot{k}(t) + \frac{g_q}{1-\alpha} k(t) + g_A k(t) + g_L k(t) = k(t)^{\alpha} - c(t) - \delta k(t).$$

So $\dot{k}(t) = k(t)^{\alpha} - c(t) - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L \right) k(t)$

- Hamiltonian:

$$H = \frac{[c(t)]^{1-\sigma}}{1-\sigma} + \lambda(t)\left[k(t)^\alpha - c(t) - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L\right)k(t)\right].$$

FOCs are

$$\begin{aligned} H_c &= 0 \\ H_k &= \eta\lambda(t) - \dot{\lambda}(t) \end{aligned}$$

- So

$$\begin{aligned} c(t)^{-\sigma} - \lambda(t) &= 0, \\ \text{and } \lambda(t)\left[\alpha k(t)^{\alpha-1} - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L\right)\right] &= \eta\lambda(t) - \dot{\lambda}(t). \\ \text{So } \alpha k(t)^{\alpha-1} - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L\right) &= \eta - \frac{\dot{\lambda}(t)}{\lambda(t)} \\ &= \eta + \sigma\frac{\dot{c}(t)}{c(t)} \end{aligned}$$

- Done! Let's recover our original notation:

$$\begin{aligned} \alpha k(t)^{\alpha-1} - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L\right) &= \eta + \sigma\frac{\dot{c}(t)}{c(t)} \\ \alpha\left(\frac{K(t)}{q(t)^{1/(1-\alpha)}A(t)L(t)}\right)^{\alpha-1} - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L\right) &= \rho - g_q[(1-\sigma)\alpha/(1-\alpha)] - g_A(1-\sigma) - g_L \\ &\quad + \sigma\left(\frac{\dot{C}(t)}{C(t)} - \frac{\alpha}{1-\alpha}\underbrace{\frac{\dot{q}(t)}{q(t)}}_{g_q} - \underbrace{\frac{\dot{A}(t)}{A(t)}}_{g_A} - \underbrace{\frac{\dot{L}(t)}{L(t)}}_{g_L}\right) \\ \alpha\left(\frac{K(t)}{q(t)^{1/(1-\alpha)}A(t)L(t)}\right)^{\alpha-1} - \left(\delta + \frac{g_q}{1-\alpha} + g_A + g_L\right) &= \rho - \frac{(1-\sigma)\alpha}{1-\alpha}g_q - g_A(1-\sigma) - g_L \\ &\quad + \sigma\frac{\dot{C}(t)}{C(t)} - \sigma\frac{\alpha}{1-\alpha}g_q - \sigma g_A - \sigma g_L \\ \alpha\left(\frac{K(t)}{q(t)^{1/(1-\alpha)}A(t)L(t)}\right)^{\alpha-1} - \left(\delta + \frac{g_q}{1-\alpha}\right) &= \rho - \frac{\alpha}{1-\alpha}g_q + \sigma\frac{\dot{C}(t)}{C(t)} - \sigma g_L \\ \alpha\left(\frac{K(t)}{q(t)^{1/(1-\alpha)}A(t)L(t)}\right)^{\alpha-1} - \delta &= \rho + g_q + \sigma\left(\frac{\dot{C}(t)}{C(t)} - g_L\right) \end{aligned}$$

- Note: If $q(t) = 1$ (i.e., $g_q = 0$), then this becomes exactly the same as the previous model.
- Note: The first term of the LHS is, again, the interest rate!
- Rewriting this FOC and all constaints:

$$(1)\ \alpha\left(\frac{K(t)}{q(t)^{1/(1-\alpha)}A(t)L(t)}\right)^{\alpha-1} - \delta = \rho + g_q + \sigma\left(\frac{\dot{C}(t)}{C(t)} - g_L\right)$$

$$(2)\ Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$$

$$(3)\ \dot{K}(t) = q(t)I(t) - \delta K(t)$$

$$(4)\ Y(t) = C(t) + I(t)$$

- Consider the BGP in which all variables grow at constant rates.
- In (1), $\frac{K(t)}{q(t)^{1/(1-\alpha)}A(t)L(t)}$ is constant, implying that

$$(\text{A})\ g_K = \frac{1}{1-\alpha}g_q + g_A + g_L$$

- In (2),

$$g_Y = \alpha g_K + (1-\alpha)g_A + (1-\alpha)g_L$$

So these two results imply

$$\begin{aligned}(\text{B})\ g_Y &= \alpha g_K + (1-\alpha)g_A + (1-\alpha)g_L \\ &= \alpha\left(\frac{1}{1-\alpha}g_q + g_A + g_L\right) + (1-\alpha)g_A + (1-\alpha)g_L \\ &= \frac{\alpha}{1-\alpha}g_q + g_A + g_L\end{aligned}$$

So $\boxed{g_Y \text{ is different from } g_K}$! That is,

$$\begin{aligned}g_Y &= \frac{\alpha}{1-\alpha}g_q + g_A + g_L \\ &= -g_q + \underbrace{\frac{1}{1-\alpha}g_q + g_A + g_L}_{=g_K} \\ &= g_K - g_q\end{aligned}$$

- In (3),

$$\begin{aligned}\frac{\dot{K}(t)}{K(t)} &= q(t)\frac{I(t)}{Y(t)}\frac{Y(t)}{K(t)} - \delta \\ &= s(t)q(t)\frac{Y(t)}{K(t)} - \delta\end{aligned}$$

  Notice that $q(t)\frac{Y(t)}{K(t)}$ is constant because we know $g_Y = g_K - g_q$. So $s(t)$ is constant!

- **Calibration**: Use the following data
- $s = 0.20$ (same as before)
- $g_L = 0.011$ (same as before)
- $g_Y = g_C = 0.033$ (same as before)
- $\delta_K(K/Y) = 0.12$ (same as before)
- $\alpha = 0.32$ (same as before)
- $q(t)$ is the units of capital goods that can be bought by one unit of physical good. = (physical good price) / (capital good price)
- So g_q = (growth rate of physical good price) − (growth rate of capital good price) =4.0% − 2.9% = $\boxed{1.1\%}$
- Write the above equations on BGP:
- (B):

$$\begin{aligned} g_Y &= \frac{\alpha}{1-\alpha}g_q + g_A + g_L \\ 0.033 &= \frac{0.32}{1-0.32}0.011 + g_A + 0.011 \\ \text{So } g_A &= 0.017 \end{aligned}$$

- (A):

$$\begin{aligned} g_K &= g_Y + g_q \\ &= 0.033 + 0.011 = 0.044 \end{aligned}$$

- We can go further as usual to calibrate δ, r, etc.

- For growth accounting, this is enough.
- **Method 2**:
- $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$
  $\Longrightarrow\ Y_t^{1-\alpha} = (K_t/Y_t)^{\alpha}(A_t L_t)^{1-\alpha}$
  $\Longrightarrow\ (Y_t/L_t)^{1-\alpha} = (K_t/Y_t)^{\alpha} {A_t}^{1-\alpha}$
  $\Longrightarrow\ Y_t/L_t = (K_t/Y_t)^{\alpha/(1-\alpha)} A_t$

$$\Longrightarrow \underbrace{g_{Y/L}}_{\substack{(A)\text{ Real Per-capita GDP growth} \\ = 2.2\%}} = \underbrace{\frac{\alpha}{1-\alpha} g_{K/Y}}_{\substack{(B)\text{ K/Y's contirubtion} \\ = \frac{\alpha}{1-\alpha}(g_K - g_Y) = 0.5\% \\ \text{(contribution from cheaper} \\ \text{capital goods)}}} + \underbrace{g_A}_{\substack{(C)\text{ A's contribution} \\ = \text{remaining } 1.7\%}}$$

- Contribution from (B): $\frac{0.5}{2.2} = 23\%$. (It was zero previously.)
- So what causes capital goods to become cheaper?
  1. R&D
  2. Importing (cheaper) capital goods from foreign economies (Eaton and Kortum, 2001)

EXERCISES

1. References:

   Greenwood, J., Z. Hercowitz and P. Krusell (1997), "Long-Run Implications of Investment-Specific Technological Change," American Economic Review, 87(3), 342-362.

   Eaton, J., and S. Kortum (2001), "Trade in Capital Goods," European Economic Review, 45(7), 1195-1235.

   In class, we discussed Greenwood, Hercowitz and Krusell (1997) to understand the contribution of a price decline in capital goods to economic growth. In this problem, you will follow Eaton and Korum (2001) to study the role of importing capital goods.

   Time is discrete. The world consists of two economies, "North" and "South". "North" produces both types of goods, consumption (C) goods and capital (K) goods. "South" produces C goods only. There is "North-South" trade, in which "North" exports only K goods to "South" and imports only C goods from "South". Of course, "South" exports only C goods and imports only K goods. Notice that "North" may still produce C goods. That is, C consumption in "North" equals C production in "North" plus C imports from "South".

   "North" has L workers. "South" has L^* workers. There is no population growth in either economy. All markets are perfectly competitive. Further descriptions follow:

   (1) $L^* = L^*_C$: In "South", all L^* workers work in C industry.

   (2) $L = L_{Ct} + L_{Kt}$: In "North" at period t, out of L workers, L_{Ct} workers work in C industry, and the remaining L_{Kt} workers work in K industry.

   (3) $K^*_t = K^*_{Ct}$: In "South", all K goods are devoted to C industry. K^*_t is physical capital stock. K^*_{Ct} is the physical capital stock devoted to C industry.

   (4) $K_t = K_{Ct} + K_{Kt}$: In "North", out of K_t units of K stock, K_{Ct} units are devoted to C industry, and the remaining K_{Kt} units are devoted to K industry.

   (5) $Q^*_{Ct} = B^* \left(L^*_C\right)^{1-\alpha} \left(K^*_{Ct}\right)^{\alpha} = B^* \left(L^*\right)^{1-\alpha} \left(K^*_t\right)^{\alpha}$: Production function of C goods in "South". Q^*_{Ct} is output. B^* is constant TFP.

   (6) $P^*_C = 1$: P^*_C is the price ($) of C goods in "South" is normalized to 1.

   (7) $Y^*_t = P^*_C Q^*_{Ct} = B^* \left(L^*\right)^{1-\alpha} \left(K^*_t\right)^{\alpha}$: Y^*_t is "South" income ($).

   (8) $Q_{Ct} = L^{1-\alpha}_{Ct} K^{\alpha}_{Ct}$: Production function of C goods in "North". "North" TFP is normalized to 1 (i.e., $B = 1$).

(9) $d_C > 1$: d_C units of C goods need to be shipped from "South" so that 1 unit of C good arrives in "North". That is, d_C reflects the cost of international trade, including tariffs and transport costs.

(10) $P_C = d_C P_C^* = d_C$: P_C is the price ($) of C goods in "North". To understand this, notice that in equilibrium, C producers in "South" find (i) selling C goods domestically and (ii) exporting them indifferent. By selling 1 unit of C good domestically, a producer receives $P_C^* = 1$. By exporting 1 unit of C good, only $1/d_C$ units arrive in "North", so she receives P_C/d_C. So in equilibrium, $P_C^* = 1 = P_C/d_C$.

(11) $Q_{Kt} = A_t L_{Kt}^{1-\alpha} K_{Kt}^{\alpha}$: Production function of K goods in "North". Q_{Kt} is K industry output. A_t is productivity. Notice that α is common in K industry and C industry.

(12) $A_{t+1} = (1 + g_A)A_t$: Productivity growth in K industry in "North". A_t is the productivity, exogenously growing.

(13) $P_{Kt} = P_C/A_t = d_C/A_t$: Price ($) of K goods in "North". To understand this, notice that K and L are allocated between the industries according to an equalization of values of marginal products. The values of marginal products of labor in the two sectors are

$$P_C(1-\alpha)L_{Ct}^{-\alpha}K_{Ct}^{\alpha} = P_{Kt}A_t(1-\alpha)L_{Kt}^{-\alpha}K_{Kt}^{\alpha}.$$

Rearranging,

$$P_C\left(K_{Ct}/L_{Ct}\right)^{\alpha} = P_{Kt}A_t\left(K_{Kt}/L_{Kt}\right)^{\alpha}.$$

Similarly, the values of marginal products of capital in the two sectors are

$$P_C\alpha L_{Ct}^{1-\alpha}K_{Ct}^{\alpha-1} = P_{Kt}A_t\alpha L_{Kt}^{1-\alpha}K_{Kt}^{\alpha-1}.$$

Rearranging,

$$P_C(K_{Ct}/L_{Ct})^{\alpha-1} = P_{Kt}A_t(K_{Kt}/L_{Kt})^{\alpha-1}.$$

These two conditions imply $K_{Ct}/L_{Ct} = K_{Kt}/L_{Kt}$, as well as $P_C = P_{Kt}A_t$.

(14) $d_K > 1$: d_K units of K goods need to be shipped from "North" so that 1 unit of K good arrives in "South".

(15) $P_{Kt}^* = d_K P_{Kt} = d_K d_C/A_t$: Price ($) of K goods in "North".

(16) $Y_t = P_C Q_{Ct} + P_{Kt}Q_{Kt} = d_C L_{Ct}^{1-\alpha}K_{Ct}^{\alpha} + (d_C/A_t)A_t L_{Kt}^{1-\alpha}K_{Kt}^{\alpha} = d_C(L_{Ct}^{1-\alpha}K_{Ct}^{\alpha} + L_{Kt}^{1-\alpha}K_{Kt}^{\alpha}) = d_C L^{1-\alpha}K_t^{\alpha}$: "North" income ($). The last equality follows from the fact that $K_{Ct}/L_{Ct} = K_{Kt}/L_{Kt}$. That is,

$$\begin{aligned}L_{Ct}^{1-\alpha}K_{Ct}^{\alpha} + L_{Kt}^{1-\alpha}K_{Kt}^{\alpha} &= L_{Ct}(K_{Ct}/L_{Ct})^{\alpha} + L_{Kt}(K_{Kt}/L_{Kt})^{\alpha}\\ &= (L_{Ct} + L_{Kt})(K_t/L)^{\alpha} \text{ (since } K_{Ct}/L_{Ct} = K_{Kt}/L_{Kt} = K_t/L)\\ &= L(K_t/L)^{\alpha} = L^{1-\alpha}K_t^{\alpha}\end{aligned}$$

(17) $K_{t+1}^* = (1-\delta)K_t^* + I_t^*$: "South" law of motion of physical capital, where I_t^* is the units of K goods invested.

(18) $I_t^* = s^* Y_t^* / P_{Kt}^* = s^* B^* (L^*)^{1-\alpha} (K_t^*)^\alpha / (d_K d_C / A_t) = s^* A_t B^* (L^*)^{1-\alpha} (K_t^*)^\alpha / (d_K d_C)$: "South" spends a constant fraction s^* of its income ($) to accumulate K goods.

(19) $K_{t+1} = (1-\delta)K_t + I_t$: "North" law of motion of physical capital, where I_t is the units of K goods invested.

(20) $I_t = sY_t/P_{Kt} = s(d_C L^{1-\alpha} K_t^\alpha)/(d_C/A_t) = sA_t L^{1-\alpha} K_t^\alpha$: "North" spends a constant fraction s of its income ($) to buy K goods.

(21) (You do not need this part for the questions, but I discuss anyway.) Market Clears: "South" imports I_t^* units of K goods, implying that "North" ships $d_K I_t^*$ units of K goods to export to "South." In addition, "North"'s domestic demand of K goods is I_t units. This implies that "North" produces $Q_{Kt} = d_K I_t^* + I_t$ units. Rearranging,

$$\begin{aligned} A_t L_{Kt}^{1-\alpha} K_{Kt}^\alpha &= d_K s^* A_t B^* (L^*)^{1-\alpha} (K_t^*)^\alpha / (d_K d_C) + sA_t L^{1-\alpha} K_t^\alpha \\ &= s^* A_t B^* (L^*)^{1-\alpha} (K_t^*)^\alpha / d_C + sA_t L^{1-\alpha} K_t^\alpha \end{aligned}$$

Dividing both sides by A_t and using $K_{Ct}/L_{Ct} = K_{Kt}/L_{Kt} = K_t/L$,

$$L_{Kt}(K_t/L) = s^* B^* (L^*)^{1-\alpha} (K_t^*)^\alpha / d_C + sL^{1-\alpha} K_t^\alpha$$

Equivalently,

$$\frac{L_{Kt}}{L} = \frac{s^* B^* (L^*)^{1-\alpha} (K_t^*)^\alpha / d_C + sL^{1-\alpha} K_t^\alpha}{K_t}$$

This determines "North" allocation between K and C industries.

Now we study the implications on income growth.

(a) Assume that K growth in "North" is constant. What is its growth rate, g_K, in terms of paramters? What is the income growth, g_Y, in terms of paramters? What is the level of $\frac{A_t}{K_t^{1-\alpha}}$ at any period t in which K growth is constant, in terms of paramters? Fill the blank with a function of parameters:

$$\frac{Y_t}{L} = \boxed{} \times (A_t)^{\frac{\alpha}{1-\alpha}}$$

Answer: *(19) and (20) imply*

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ &= (1-\delta)K_t + sA_t L^{1-\alpha} K_t^\alpha \end{aligned}$$

So

$$\frac{K_{t+1}}{K_t} = (1-\delta) + s\frac{A_t}{K_t^{1-\alpha}}L^{1-\alpha}$$

$\frac{K_{t+1}}{K_t}$ *will be constant if and only if* $g_A = (1-\alpha)g_K$, *or equivalently,*

$$g_K = \frac{g_A}{1-\alpha}.$$

From (16): $Y_t = d_C L^{1-\alpha} K_t^{\alpha}$, *we know* $g_Y = \alpha g_K$. *Hence,*

$$g_Y = \alpha g_K = \frac{\alpha g_A}{1-\alpha}.$$

Then,

$$\underbrace{1+g_K}_{=1+\frac{g_A}{1-\alpha}} = (1-\delta) + s\frac{A_t}{K_t^{1-\alpha}}L^{1-\alpha}$$

Hence,

$$\frac{A_t}{K_t^{1-\alpha}} = \frac{\frac{g_A}{1-\alpha}+\delta}{sL^{1-\alpha}}.$$

Hence, from (16) again,

$$\begin{aligned}\frac{Y_t}{L} &= d_C L^{-\alpha} K_t^{\alpha} = d_C L^{-\alpha}(K_t^{1-\alpha})^{\frac{\alpha}{1-\alpha}} \\ &= d_C L^{-\alpha}\left(\frac{sL^{1-\alpha}A_t}{\frac{g_A}{1-\alpha}+\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ &= d_C\left(\frac{s}{\frac{g_A}{1-\alpha}+\delta}\right)^{\frac{\alpha}{1-\alpha}}(A_t)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

(b) Assume that K growth in "South" is also constant. What is its growth rate, g_K^*, in terms of paramters? What is the income growth, g_Y^*, in terms of paramters? What is the level of $\frac{A_t}{(K_t^*)^{1-\alpha}}$ at any period t in which K growth is constant, in terms of paramters? Fill the blank with a function of parameters:

$$\frac{Y_t^*}{L^*} = \boxed{} \times (A_t)^{\frac{\alpha}{1-\alpha}}$$

Answer: (17) and (18) imply

$$\begin{aligned}K_{t+1}^* &= (1-\delta)K_t^* + I_t^* \\ &= (1-\delta)K_t^* + s^* A_t B^* (L^*)^{1-\alpha}(K_t^*)^{\alpha}/(d_K d_C)\end{aligned}$$

So
$$\frac{K_{t+1}^*}{K_t^*} = (1-\delta) + s^* \frac{A_t}{(K_t^*)^{1-\alpha}} B^* (L^*)^{1-\alpha} / (d_K d_C)$$
$\frac{K_{t+1}^*}{K_t^*}$ *will be constant if and only if* $g_A = (1-\alpha) g_K^*$*, or equivalently,*
$$g_K^* = \frac{g_A}{1-\alpha}. \text{ (K growth rate is the same as "North".)}$$
From (7): $Y_t^* = B^* (L^*)^{1-\alpha} (K_t^*)^{\alpha}$*, we know* $g_Y^* = \alpha g_K^*$*. Hence,*
$$g_Y^* = \alpha g_K^* = \frac{\alpha g_A}{1-\alpha}. \text{ (Y growth rate is the same as "North".)}$$
Then,
$$\underbrace{1 + g_K^*}_{=1+\frac{g_A}{1-\alpha}} = (1-\delta) + s^* \frac{A_t}{(K_t^*)^{1-\alpha}} B^* (L^*)^{1-\alpha} / (d_K d_C)$$
Hence,
$$\frac{A_t}{(K_t^*)^{1-\alpha}} = (d_K d_C) \frac{\frac{g_A}{1-\alpha} + \delta}{s^* B^* (L^*)^{1-\alpha}}.$$
Hence, from (7) again,
$$\begin{aligned} \frac{Y_t^*}{L^*} &= B^* (L^*)^{-\alpha} (K_t^*)^{\alpha} = B^* (L^*)^{-\alpha} \left[(K_t^*)^{1-\alpha} \right]^{\frac{\alpha}{1-\alpha}} \\ &= B^* (L^*)^{-\alpha} \left[\frac{s^* B^* (L^*)^{1-\alpha} A_t}{d_K d_C \left(\frac{g_A}{1-\alpha} + \delta \right)} \right]^{\frac{\alpha}{1-\alpha}} \\ &= (B^*)^{\frac{1}{1-\alpha}} \left[\frac{s^*}{d_K d_C \left(\frac{g_A}{1-\alpha} + \delta \right)} \right]^{\frac{\alpha}{1-\alpha}} (A_t)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

(c) Describe "South"'s gain from international trade with "North".

Answer: *"South" does not produce any K goods. Since K accumulation is the only source of economic growth, "South" would not have any economic growth. If "South" trades with "North", then the gain is huge because now "South" grows at* $g_Y^* = 0.55\%$ *forevever.*

(d) How is P_{Kt}/P_C evolve over time? That is, does P_{Kt}/P_C increase or decrease? According to data we discussed in class, P_{Kt}/P_C decreases at 1.1% per period. What does it imply about g_A?

Answer: *From (10) and (13),*
$$\frac{P_{Kt}}{P_C} = \frac{d_C/A_t}{d_C} = \frac{1}{A_t}.$$
Hence, $g_A = 0.011$.

(e) Continue to assume your result in (d). Further, assume

$$\frac{Y_t/L_t}{Y_t^*/L_t^*} = 4,$$

as well as $\alpha = 1/3$, $\delta = 0.05$, $s = 0.2$ and $s^* = 0.4$. What do your results so far imply about other parameter values?

Answer: *From the previous subquestions,*

$$\frac{Y_t/L_t}{Y_t^*/L_t^*} = \frac{d_C\left(\frac{s}{\frac{g_A}{1-\alpha}+\delta}\right)^{\frac{\alpha}{1-\alpha}}(A_t)^{\frac{\alpha}{1-\alpha}}}{(B^*)^{\frac{1}{1-\alpha}}\left[\frac{s^*}{d_K d_C\left(\frac{g_A}{1-\alpha}+\delta\right)}\right]^{\frac{\alpha}{1-\alpha}}(A_t)^{\frac{\alpha}{1-\alpha}}}$$

$$= \frac{d_C\overbrace{\left(\frac{0.2}{\frac{0.011}{1-1/3}+0.05}\right)^{1/2}}^{=1.7342}}{(B^*)^{\frac{1}{1-1/3}}\left[\frac{1}{d_K d_C}\right]^{1/2}\underbrace{\left[\frac{0.4}{\left(\frac{0.011}{1-1/3}+0.05\right)}\right]^{1/2}}_{=2.4526}}$$

$$= \frac{(d_C)^{3/2}(d_K)^{1/2}}{(B^*)^{3/2}}0.70709 = 4$$

Hence,

$$\frac{(d_C)^3 d_K}{(B^*)^3} = \left(\frac{4}{0.70709}\right)^2 = 32.002.$$

(f) Compare P_{Kt}^*/P_C^* and P_{Kt}/P_C. In which economy the relative price of K good to C good higher?

Answer: *We have*

$$\frac{P_{Kt}}{P_C} = \frac{1}{A_t}$$

and

$$\frac{P_{Kt}^*}{P_C^*} = d_K P_{Kt} = d_K P_C/A_t = d_K d_C \frac{1}{A_t}.$$

Since $d_K d_C > 1$, $\frac{P_{Kt}^*}{P_C^*}$ *is higher.*

2. Consider a closed economy. There are two types of goods: consumption goods and capital goods. The economy has a fixed size of population (normalized to one).

Consumption goods are produced as

$$Y(t) = A(t)K(t)^{\alpha},$$
$$A(t) = A(0)e^{g_A t},$$

for all $t > 0$, and for $0 < \alpha < 1$ and $g_A > 0$, given $Y(0)$, $K(0)$ and $A(0)$, where $Y(t)$ is the output in units of consumption goods at period t, $A(t)$ is exogenously growing productivity, and $K(t)$ is the capital stock in units of capital goods. Here, $K(t)$ evolves as

$$\dot{K}(t) = q(t)I(t) - \delta K(t), \tag{1}$$

for $0 < \delta < 1$, where $\dot{K}(t) \equiv dK(t)/dt$, $I(t)$ is the units of consumption goods to be used to produce capital goods, and $q(t)$ is the exogenously growing productivity for capital goods production, in which $I(t)$ units of consumption goods are transformed into $q(t)I(t)$ units of capital goods. Here,

$$q(t) = q(0)e^{g_q t},$$

for $g_q > 0$, given $q(0)$. We also write

$$I(t) = s(t)Y(t),$$

where $0 < s(t) < 1$ is interpreted as an investment rate (and, of course, the remaining $Y(t) - I(t)$ units of consumption goods are consumed).

Throughout this problem, make reasonable assumptions if required. Subproblems are equally graded.

(a) Consider a balanced growth path in which all variables grow at constant rates and $s(t)$ is constant. Denote by g_X the growth rate of a variable $X(t)$ on this balanced growth path. Fill the blanks (A) and (B) with some functions only of parameter α (like $1/(1+\alpha)$, α^2, etc.), in the following growth decomposition:

$$g_Y = \boxed{\text{(A)}} \times g_A + \boxed{\text{(B)}} \times g_q.$$

If you know the values of α, g_A and g_q, then what can be inferred from this growth decomposition?

Answer: *From* $Y(t) = A(t)K(t)^{\alpha}$, *we have* $Y(t)^{1-\alpha} = A(t)K(t)^{\alpha}/Y(t)^{\alpha}$. *Hence,*

$$(1-\alpha)g_Y = g_A + \alpha g_{K/Y}.$$

So

$$g_Y = \frac{1}{1-\alpha}g_A + \frac{\alpha}{1-\alpha}g_{K/Y}.$$

Now from capital accumulation,

$$\frac{\dot{K}(t)}{K(t)} = q(t)\frac{I(t)}{Y(t)}\frac{Y(t)}{K(t)} - \delta$$

So

$$g_K = s\left[q(t)\frac{Y(t)}{K(t)}\right] - \delta$$

For this equation to hold, we should have $g_q = g_{K/Y}$. *Therefore,*

$$g_Y = \frac{1}{1-\alpha}g_A + \frac{\alpha}{1-\alpha}g_q.$$

So $\boxed{\text{(A)}} = \frac{1}{1-\alpha}g_A$ *and* $\boxed{\text{(B)}} = \frac{\alpha}{1-\alpha}g_q$. *Out of output growth* g_Y, *a size of* $\frac{\alpha}{1-\alpha}g_q$ *is from an improvement in capital goods production (such as cheaper computers, cheaper machines, etc.)*

(b) Now suppose that the economy is open to international trade. This economy is small, and hence, it takes international prices as given. Dollar is a single international currency. The international price of consumption good is $p_C(t)$ dollars at period t. The international price of capital good is $p_K(t)$ dollars at period t. The economy can buy or sell any amounts of goods at these international prices as long as trade is balanced (i.e., trade deficit is zero) at each period t. There are no tariffs or other distortions in trade.

Assume $q(t) < p_C(t)/p_K(t)$. Which good does this economy import at period t? Which good does this economy export?

Answer: *Suppose that this economy has produced* 1 *unit of consumption good. This can be (i) transformed into* $q(t)$ *units of capital goods using domestic technology, or (ii) sold at international market at* $p_C(t)$ *dollars, which can purchase* $p_C(t)/p_K(t)$ *units of capital goods at international market. To conclude,*

(A) If $\boxed{q(t) < p_C(t)/p_K(t)}$, *then this economy does not produce any capital goods. That is, it specializes in consumption goods, exports them, and import capital goods.*

(B) If $\boxed{q(t) > p_C(t)/p_K(t)}$, *then the opposite: This economy transforms all production into capital goods. So it specializes in capital goods, exports them, and imports consumption goods.*

(c) Continue to consider the same economy (considered in (b) with $q(t) < p_C(t)/p_K(t)$). Provide a small-open-economy version of equation (1).

Hint: Assume that this economy sells all its output (whether it is consumption goods or capital goods) at international market, and then spends a *given* fraction $1 - s$ of the dollars received to purchase consumption goods and the remaining fraction s to purchase capital goods at international market. Here, s is the investment rate which has the same value under autarky or with trade.

Answer: *The output, $Y(t)$ units of consumption goods, is sold at international market, which brings $p_C(t)Y(t)$ dollars. Since a fraction s of these dollars are spent to accumulate physical capital, the economy can purchase $sY(t)p_C(t)/p_K(t)$ units of capital goods. Hence,*

$$\dot{K}(t) = sY(t)p_C(t)/p_K(t) - \delta K(t).$$

That is, $sY(t)p_C(t)/p_K(t)$ replaces $q(t)I(t)$.

(d) Continue to consider the same economy. Provide a growth decomposition for this economy (just as you did in (a)).

Answer: *From capital accumulation,*

$$\frac{\dot{K}(t)}{K(t)} = s\frac{p_C(t)}{p_K(t)}\frac{Y(t)}{K(t)} - \delta$$

So

$$g_K = s\left[\frac{p_C(t)}{p_K(t)}\frac{Y(t)}{K(t)}\right] - \delta$$

For this equation to hold, we should have $g_{p_C/p_K} = g_{K/Y}$ where g_{p_C/p_K} is the growth of international relative price, $\frac{p_C(t)}{p_K(t)}$. Therefore, the growth decomposition becomes

$$g_Y = \frac{1}{1-\alpha}g_A + \frac{\alpha}{1-\alpha}g_{p_C/p_K}.$$

(e) Discuss the gain from international trade using your results so far.

Answer: *By opening to international trade, more capital goods are accumulated ($sY(t)\frac{p_C(t)}{p_K(t)} > sq(t)Y(t)$).*

Comparing (a) and (d), the growth rate of an open economy can be different from the growth rate of a closed economy.

Note: Up to here is enough for about one point out of two possible points. An ambitious student can continue the argument, for example, as follows.

- *If $g_{p_C/p_K} > g_q$, then this gap (an increase in the growth rate) is the gain in long-run economic growth. If $g_{p_C/p_K} < g_q$, then the growth even decelerates, but there still is a static gain coming from cheaper capital goods, and eventually the economy will move over to the other situation in which $q(t) < p_C(t)/p_K(t)$.*
- *This table will help to understand static and dynamic gains of international trade.*

	Autarky	*Situation (A)* $(q(t) < \frac{p_C(t)}{p_K(t)})$	*Situation (B)* $(q(t) > \frac{p_C(t)}{p_K(t)})$
Initial production of C	$Y(t) = A(t)K(t)^{\alpha}$	$Y(t) = A(t)K(t)^{\alpha}$	$Y(t) = A(t)K(t)^{\alpha}$
Production of K	$q(t)sY(t)$	*0*	$q(t)Y(t)$
Selling at international market	*0*	$Y(t)$ *units of C*	$q(t)Y(t)$ *units of K*
Dollars made	*0*	$p_C(t)Y(t)$	$p_K(t)q(t)Y(t)$
Dollars spent on K purchase	*0*	$sp_C(t)Y(t)$	$sp_K(t)q(t)Y(t)$
Units of K imported	*0*	$sY(t)\frac{p_C(t)}{p_K(t)}$	$sq(t)Y(t)$
Dollars spent on C purchase	*0*	$(1-s)p_C(t)Y(t)$	$(1-s)p_K(t)q(t)Y(t$
Units of C imported	*0*	$(1-s)Y(t)$	$(1-s)q(t)Y(t)\frac{p_K(t}{p_C(t}$
Units of C consumed	$(1-s)Y(t)$	$(1-s)Y(t)$	$(1-s)q(t)Y(t)\frac{p_K(t}{p_C(t}$
Gain here?		*No. Same as Autarky.*	*More C consumed since* $q(t)\frac{p_K(t)}{p_C(t)} > 1.$
Units of K accumulated	$sq(t)Y(t)$	$sY(t)\frac{p_C(t)}{p_K(t)}$	$sq(t)Y(t)$
Gain here?		*More K goods accumulated since* $q(t) < \frac{p_C(t)}{p_K(t)}$	*No. same as Autark*
Gain from trade in dollars	*0*	$p_K(t)sY(t) \times \left(\frac{p_C(t)}{p_K(t)} - q(t)\right)$	$p_C(t)(1-s)Y(t) \times \left[\frac{p_K(t)}{p_C(t)}q(t) - 1\right]$

- *Situation (B): The level effect is that more consumption goods are consumed. It does not affect the capital accumulation, so the formula in (a) is not affected at all.*
- *All these discussions are under the assumption that $s(t)$ remains the same. Depending on the utility function, this economy will perhaps increase $s(t)$ to accumulate more capital goods because now more consumption goods are available.*

Human Capital Accumulation and Growth

References

- Hall, R. E., and C. I. Jones (1999), "Why Do Some Countries Produce So Much More Output Per Worker Than Others?", Quarterly Journal of Economics, 114(1), 83-116.
- Choi, S. M. (2011), "How Large are Learning Externalities?," International Economic Review, 52(4), 1077-1103.

- **Data**: "Education" in EconS 427
  - In XS of workers, more education increases the earnings. (1 year $\uparrow \longrightarrow$ 10% $\uparrow$) This is the **causality**. (Empirical labor literature seems to have the consensus.)
  - A positive correlation between national income and education index
- **Data**: Barro and Lee, http://www.cid.harvard.edu/ciddata/ciddata.html, "Appendix Data Tables"

1. Measuring the Human Capital

- Empirical Labor: $\boxed{\text{1 year} \uparrow \text{in school} \longrightarrow \text{Wage 10\%} \uparrow}$
  from **Mincer Regression**: $\log(\text{wage}_i) = \alpha + \beta\ \text{age}_i + \gamma\ (\text{years of schooling})_i + ...$
- How can we incorporate this in macro?
- **Assumption**: Human capital is accumulated by schooling.
- $Y(t) = K(t)^\alpha (A(t)h(t)L(t))^{1-\alpha}$
  $h(t)$: human capital per worker
  $A(t)$: everything else
- Each input receives the marginal product.

- For example,

  1 unit of K —— after 1 period ——> Original 1 unit is returned

  But δ units depreciates

  Interest $r(t)$ is paid as compensation

$$\begin{aligned} r(t) &= \text{MPK}(t) \\ &= \alpha K(t)^{\alpha-1}(A(t)h(t)L(t))^{1-\alpha} \end{aligned}$$

- Similarly,

  1 unit of human capital – after 1 period –> Original 1 unit remains

  Some fraction may depreciate (Forgetting?)

  wage rate $w(t)$ is paid as compensation

$$\begin{aligned} w(t) &= \text{MPH}(t) \\ &= (1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}(h(t)L(t))^{-\alpha} \end{aligned}$$

- If Worker 1 has 1 unit of human capital, he earns $w(t)$.

  If Worker 2 has 10 units, he earns $10w(t)$.

- What about

$$h(S) = e^{0.1S},$$

  where $S =$ (years of schooling)?

- Then $\frac{dh(S)}{dS} = \frac{de^{0.1S}}{dS} = 0.1e^{0.1S} = 0.1h(S)$.
- **Assumption**: Every worker is identical (with same S).
- According to Barro-Lee data, average schooling years for the people at ages of 25 or above:

  1960: $\boxed{\text{8.66 years}}$ $\Longrightarrow$ 2000: $\boxed{\text{12.25 years}}$

  (Applying 1995-2000 growth rate to 2000-2005, 2005: 12.32.)

  So $h_{1960} = e^{0.1\times 8.66}$ and $h_{2005} = e^{0.1\times 12.32}$.

- $g_h = \frac{\log(h_{2005})-\log(h_{1960})}{2005-1960} = \frac{\log(e^{0.1\times 12.25})-\log(e^{0.1\times 8.66})}{2005-1960} = \frac{0.1\times 12.25-0.1\times 8.66}{2005-1960} = \boxed{0.8\%}$.
- Growth accounting:
- **Method 1**:

- $Y_t = K_t^{\alpha}(A_t h_t L_t)^{1-\alpha}$

$$\implies Y_t/L_t = (K_t/L_t)^{\alpha} A_t^{1-\alpha} h_t^{1-\alpha}$$

$$\implies \underbrace{g_{Y/L}}_{\substack{(A)\text{ Real Per-capita GDP growth} \\ =2.2\%}} = \underbrace{\alpha g_{K/L}}_{(B)\text{ K/L's contirubtion}} + \underbrace{(1-\alpha)g_A}_{(C)\text{ A's contribution}} + \underbrace{(1-\alpha)g_h}_{\substack{(D)\text{ h's contribution} \\ =0.5\%}}$$

- Contribution from (D): $\frac{0.5}{2.2} = 23\%$

- **Method 2**:

- $Y_t = K_t^{\alpha}(A_t h_t L_t)^{1-\alpha}$

$$\implies Y_t^{1-\alpha} = (K_t/Y_t)^{\alpha}(A_t h_t L_t)^{1-\alpha}$$

$$\implies (Y_t/L_t)^{1-\alpha} = (K_t/Y_t)^{\alpha} A_t^{1-\alpha} h_t^{1-\alpha}$$

$$\implies Y_t/L_t = (K_t/Y_t)^{\alpha/(1-\alpha)} A_t h_t$$

$$\implies \underbrace{g_{Y/L}}_{\substack{(A)\text{ Real Per-capita GDP growth} \\ =2.2\%}} = \underbrace{\frac{\alpha}{1-\alpha} g_{K/Y}}_{(B)\text{ K/Y's contirubtion}} + \underbrace{g_A}_{(C)\text{ A's contribution}} + \underbrace{g_h}_{\substack{(D)\text{ h's contribution} \\ =0.8\%}}$$

- Contribution from (D): $\frac{0.8}{2.2} = 36\%$

- **Problems**:

  1. Is it justified to use the cross-sectional result for time series application, assuming everyone is the same?
  2. Maybe A↑ is causing h↑ (just like A↑ is causing K/L↑)?
  3. Other sources of human capital accumulation: On-the-job training? In-home training?
  4. Human capital externalities: I become more productive if surrounded by smarter workers ("static externalities"). I learn faster if surrounded by smarter workers ("learning externalities"). Externalities are not included in the discussions so far.

- Need: A more sophisticated model.

2. Calibration of a Two-Sector Model with Human Capital Accumulation

- Contents:
  - A. Model Description
  - B. First-Order Conditions
  - C. Balanced Growth Path
  - D. Elimination of $\lambda(t)$ and $\mu(t)$
  - E. Recovery of Original Notations
  - F. Calibration

- **A. Model Description**

- In the previous model, the consumer decides $\{s(t)\}$ (investment rate) (or equivalently, $\{I(t)\}$) which determines $\{K(t)\}$.

- Can we analyze in the same way for human capital?

- Consumer decides $\{u(t)\}$ (fraction of time devoted to human capital accumulation such as schooling, R&D, in-home training, OJT, ...) which determines $\{h(t)\}$.

- Model:

$$\max_{\{C(t),I(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t)dt \text{ (same)}$$

  s.t.

$$\begin{aligned}
Y(t) &= C(t) + I(t), \text{ (same)} \\
Y(t) &= K(t)^{\alpha}(A(t) \underbrace{(1-u(t))}_{\text{only this fraction of } h(t) \text{ is used for Y production}} h(t)L(t))^{1-\alpha}, \\
A(t) &= A(0)e^{g_A t}, \text{ (same)} \\
L(t) &= L(0)e^{g_L t}, \text{ (same)} \\
\dot{K}(t) &= I(t) - \delta_K K(t), \text{ (same)}
\end{aligned}$$

- We now add a new constraint.

- For $H(t) \equiv h(t)L(t)$,

$$\dot{H}(t) = B \underbrace{u(t)}_{\text{This fraction is for H production}} H(t) - \delta_h H(t).$$

Note: (a) Human capital is the only input. (Physical input, such as school buildings, is disregarded. It is about 10% of total educational expenditure.)

(b) CRS. (To easily get the BGP.)

- Write this constraint with $h(t)$:

$$\dot{h}(t)L(t) + h(t)\dot{L}(t) = Bu(t)h(t)L(t) - \delta_h h(t)L(t).$$

$$\text{So } \dot{h}(t) + h(t)\frac{\dot{L}(t)}{L(t)} = Bu(t)h(t) - \delta_h h(t).$$

$$\text{So } \dot{h}(t) = Bu(t)h(t) - (\delta_h + g_L)h(t).$$

- Eliminate $Y(t)$ and $I(t)$,

$$\max_{\{C(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{(C(t)/L(t))^{1-\sigma}}{1-\sigma}L(t)dt$$

s.t.

$$\begin{aligned}
A(t) &= A(0)e^{g_A t}, \\
L(t) &= L(0)e^{g_L t}, \\
\dot{K}(t) &= K(t)^{\alpha}(A(t)(1-u(t))h(t)L(t))^{1-\alpha} - C(t) - \delta K(t), \\
\dot{h}(t) &= Bu(t)h(t) - (\delta_h + g_L)h(t).
\end{aligned}$$

- Define:

$$c(t) = \frac{C(t)}{A(t)L(t)},$$

$$k(t) = \frac{K(t)}{A(t)L(t)}.$$

- Then we have a simple form (same as before):

$$\begin{aligned}
&\max_{\{c(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{[c(t)A(t)]^{1-\sigma}}{1-\sigma} L(t) dt \\
&= \max_{\{c(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} \underbrace{[A(0)]^{1-\sigma}}_{\text{just constant}} e^{g_A(1-\sigma)t} \quad \underbrace{L(0)}_{\text{just constant}} \quad e^{g_L t} dt \\
&\Longrightarrow \max_{\{c(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho - g_A(1-\sigma) - g_L)t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt \\
&= \max_{\{c(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\eta t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt
\end{aligned}$$

where $\eta \equiv \rho - g_A(1-\sigma) - g_L$,

s.t. the following two constraints:

- First constraint:

$$\begin{aligned}
\frac{\dot{K}(t)}{A(t)L(t)} &= \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} ((1-u(t))h(t))^{1-\alpha} - \frac{C(t)}{A(t)L(t)} - \delta \frac{K(t)}{A(t)L(t)} \\
&= k(t)^{\alpha}((1-u(t))h(t))^{1-\alpha} - c(t) - \delta k(t)
\end{aligned}$$

We know

$$K(t) = k(t)A(t)L(t)$$

So

$$\begin{aligned}
\dot{K}(t) &= \dot{k}(t)A(t)L(t) + k(t)\dot{A}(t)L(t) + k(t)A(t)\dot{L}(t) \\
\Longrightarrow \frac{\dot{K}(t)}{A(t)L(t)} &= \dot{k}(t) + k(t)g_A + k(t)g_L
\end{aligned}$$

So the constraint becomes

$$\dot{k}(t) = k(t)^{\alpha}((1-u(t))h(t))^{1-\alpha} - c(t) - (\delta_K + g_A + g_L)k(t).$$

- Second constraint:

$$\dot{h}(t) = Bu(t)h(t) - (\delta_h + g_L)h(t). \text{ (OK)}$$

- **B. First-Order Conditions**
- How do we set up Hamiltonian when there are two constraints?

- Compare:

$\max_{\{\boxed{u(t)}\}_{t=0}^{\infty}} \int_0^\infty e^{-\rho t} h(x(t), u(t)) dt$ s.t. $\boxed{\dot{x}(t)} = g(x(t), u(t))$, $x(0)$ given. ($x(t)$: state variable, $u(t)$: control variable)	$\max_{\{\boxed{u(t), v(t)}\}_{t=0}^{\infty}} \int_0^\infty e^{-\rho t} h(x(t), y(t), u(t), v(t)) dt$ s.t. $\boxed{\dot{x}(t)} = g(x(t), y(t), u(t), v(t))$, $x(0)$ given. $\boxed{\dot{y}(t)} = f(x(t), y(t), u(t), v(t))$, $y(0)$ given. ($x(t)$, $y(t)$: state variable, $u(t)$, $v(t)$: control variable)
Hamiltonian: $H(x(t), u(t), \lambda(t)) = h(x(t), u(t))$ $+\lambda(t) g(x(t), u(t))$ ($\lambda(t)$: co-state variable) (1) $H_u = 0$ (2) $H_x = \rho\lambda(t) - \dot{\lambda}(t)$ (3) $\dot{x}(t) = g(x(t), u(t))$ (4) "transversality condition"	Hamiltonian: $H = h$ $+\lambda(t) g + \mu(t) f$ ($\lambda(t)$, $\mu(t)$: co-state variable) (1) $H_u = 0$, $H_v = 0$ (2) $H_x = \rho\lambda(t) - \dot{\lambda}(t)$, $H_y = \rho\mu(t) - \dot{\mu}(t)$ (3) $\dot{x}(t) = g$, $\dot{y}(t) = f$. (4) "transversality condition"

- Set up Hamiltonian:

$$H = \frac{[c(t)]^{1-\sigma}}{1-\sigma} + \lambda(t)\left[k(t)^\alpha((1-u(t))h(t))^{1-\alpha} - c(t) - (\delta_K + g_A + g_L)k(t)\right] + \mu(t)[Bu(t)h(t) - (\delta_h + g_L)h(t)]$$

- FOCs are

$$\begin{aligned} &(1)\ H_c = 0 \\ &(2)\ H_u = 0 \\ &(3)\ H_k = \eta\lambda(t) - \dot{\lambda}(t) \\ &(4)\ H_h = \eta\mu(t) - \dot{\mu}(t) \end{aligned}$$

- So

$$\begin{aligned} (1)\ & c(t)^{-\sigma} - \lambda(t) = 0, \\ (2)\ & -\lambda(t)k(t)^\alpha(1-\alpha)(1-u(t))^{-\alpha}h(t)^{1-\alpha} + \mu(t)Bh(t) = 0, \\ (3)\ & \lambda(t)[\alpha k(t)^{\alpha-1}((1-u(t))h(t))^{1-\alpha} - (\delta_K + g_A + g_L)] = \eta\lambda(t) - \dot{\lambda}(t), \\ (4)\ & \lambda(t)\left[k(t)^\alpha(1-u(t))^{1-\alpha}(1-\alpha)h(t)^{-\alpha}\right] + \mu(t)[Bu(t) - (\delta_h + g_L)] = \eta\mu(t) - \dot{\mu}(t), \end{aligned}$$

$$\begin{aligned} (A)\ & \dot{k}(t) = k(t)^\alpha((1-u(t))h(t))^{1-\alpha} - c(t) - (\delta_K + g_A + g_L)k(t), \\ (B)\ & \dot{h}(t) = Bu(t)h(t) - (\delta_h + g_L)h(t). \end{aligned}$$

- Before we go further, focus on the BGP.
- **C. Balanced Growth Path**
- Assume all variables grow at constant rates (or stay at constant levels).
- <Step 0: Always start with this> (1) implies

$$-\sigma\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)}.$$

$$\text{So (1') } g_c = -\frac{1}{\sigma}g_\lambda.$$

- <Step 1> (B) implies

$$\text{(B') } \frac{\dot{h}(t)}{h(t)} \equiv g_h = Bu(t) - (\delta_h + g_L)$$

So $\boxed{u \text{ is constant}}$.

- <Step 2> (3) implies

$$\text{(3') } \alpha k(t)^{\alpha-1}((1-u)h(t))^{1-\alpha} - (\delta_K + g_A + g_L) = \eta - \frac{\dot{\lambda}(t)}{\lambda(t)}$$

So $k(t)/h(t)$ should be constant, implying that $\boxed{g_k = g_h}$.

- <Step 3> (A) implies

$$\underbrace{\frac{\dot{k}(t)}{k(t)}}_{\text{const}} = ((1-u)\underbrace{\frac{h(t)}{k(t)}}_{\text{const (step 2)}})^{1-\alpha} - \frac{c(t)}{k(t)} - (\delta_K + g_A + g_L).$$

So $c(t)/k(t)$ should be constant, implying that $\boxed{g_c = g_k = g_h}$.

- <Step 4> (2) implies

$$\lambda(t)k(t)^\alpha(1-\alpha)(1-u(t))^{-\alpha}h(t)^{1-\alpha} = \mu(t)Bh(t)$$

So

$$\text{(2') } \lambda(t)\underbrace{k(t)^\alpha(1-\alpha)(1-u)^{-\alpha}h(t)^{-\alpha}}_{\text{const (}k\text{ and }h\text{ grow at same rates)}} = \mu(t)B$$

Hence, $\boxed{g_\lambda = g_\mu}$.

- **D. Elimination of $\lambda(t)$ and $\mu(t)$**
- (1), (1'):

$$(1)\ c(t)^{-\sigma} = \lambda(t), \quad (1')\ g_c = -\frac{1}{\sigma} g_\lambda.$$

Can be used to eliminate $\lambda(t)$.

- (2'):

$$(2')\ \lambda(t)k(t)^\alpha(1-\alpha)(1-u)^{-\alpha}h(t)^{-\alpha} = \mu(t)B$$

$$(2")\ \frac{1}{B}\left[\frac{k(t)}{h(t)}\right]^\alpha (1-\alpha)(1-u)^{-\alpha} = \frac{\mu(t)}{\lambda(t)}$$

- (3'):

$$(3')\ \alpha k(t)^{\alpha-1}((1-u)h(t))^{1-\alpha} - (\delta_K + g_A + g_L) = \eta - \frac{\dot{\lambda}(t)}{\lambda(t)}$$

$$\begin{aligned}(3")\ \alpha(1-u)^{1-\alpha}\left[\frac{k(t)}{h(t)}\right]^{\alpha-1} - (\delta_K + g_A + g_L) &= \eta - g_\lambda \\ &= \eta + \sigma g_c\end{aligned}$$

So λ has been eliminated.

- (4):

$$(4)\ \lambda(t)\left[k(t)^\alpha(1-u(t))^{1-\alpha}(1-\alpha)h(t)^{-\alpha}\right] + \mu(t)[Bu(t) - (\delta_h + g_L)] = \eta\mu(t) - \dot{\mu}(t)$$

$$\begin{aligned}\text{So } \frac{\lambda(t)}{\mu(t)}\left[k(t)^\alpha(1-u)^{1-\alpha}(1-\alpha)h(t)^{-\alpha}\right] + Bu - (\delta_h + g_L) &= \eta - \frac{\dot{\mu}(t)}{\mu(t)} \\ &= \eta + g_\mu \\ &= \eta + \sigma g_c\end{aligned}$$

$$\text{So } \frac{\lambda(t)}{\mu(t)}\left[\left[\frac{k(t)}{h(t)}\right]^\alpha (1-u)^{1-\alpha}(1-\alpha)\right] + Bu - (\delta_h + g_L) = \eta + \sigma g_c$$

Using (2"),

$$\frac{\left[\frac{k(t)}{h(t)}\right]^\alpha (1-u)^{1-\alpha}(1-\alpha)}{\frac{1}{B}\left[\frac{k(t)}{h(t)}\right]^\alpha (1-\alpha)(1-u)^{-\alpha}} + Bu - (\delta_h + g_L) = \eta + \sigma g_c$$

$$\text{So } B(1-u) + Bu - (\delta_h + g_L) = \eta + \sigma g_c$$

$$(4')\ B - (\delta_h + g_L) = \eta + \sigma g_c$$

So (2") and (4) are assembled, eliminating λ/μ.

- (A):

$$\dot{k}(t) = k(t)^\alpha((1-u)h(t))^{1-\alpha} - c(t) - (\delta_K + g_A + g_L)k(t)$$

$$\text{So (A') } g_k \equiv \frac{\dot{k}(t)}{k(t)} = \left[(1-u)\frac{h(t)}{k(t)}\right]^{1-\alpha} - \frac{c(t)}{k(t)} - (\delta_K + g_A + g_L)$$

- (B'):

$$\text{(B') } g_h = Bu - (\delta_h + g_L)$$

- Situation: We had 6 eqs. Eliminated λ and μ. We now have 4 eqs:

$$\text{(3") } \alpha(1-u)^{1-\alpha}\left[\frac{k(t)}{h(t)}\right]^{\alpha-1} - (\delta_K + g_A + g_L) = \eta + \sigma g_c$$

$$\text{(4') } B - (\delta_h + g_L) = \eta + \sigma g_c$$

$$\text{(A') } \left[(1-u)\frac{h(t)}{k(t)}\right]^{1-\alpha} - \frac{c(t)}{k(t)} - (\delta_K + g_A + g_L) = g_k$$

$$\text{(B') } g_h = Bu - (\delta_h + g_L)$$

- **E. Recovery of Original Notations**
- Recall:

$$\begin{aligned} c(t) &= \frac{C(t)}{A(t)L(t)}, \\ k(t) &= \frac{K(t)}{A(t)L(t)}, \\ g_c &= g_h, \\ \eta &= \rho - g_A(1-\sigma) - g_L \end{aligned}$$

- **Part 1/4**: Let's start with (3"):

$$\text{(3") } \alpha(1-u)^{1-\alpha}\left[\frac{k(t)}{h(t)}\right]^{\alpha-1} - (\delta_K + g_A + g_L) = \eta + \sigma g_c$$

$$\alpha(1-u)^{1-\alpha}\left[\frac{K(t)}{A(t)h(t)L(t)}\right]^{\alpha-1} - (\delta_K + g_A + g_L) = \rho - g_A(1-\sigma) - g_L + \sigma g_h$$

$$\alpha\left[\frac{K(t)}{A(t)(1-u)h(t)L(t)}\right]^{\alpha-1} - \delta_K = \rho + \sigma(g_A + g_h)$$

Let's look at this eq. carefully.

$$(LHS) = \underbrace{\frac{\alpha}{K(t)/Y(t)}}_{=r(t)} - \delta_K$$

- We know $\boxed{g_c = g_k = g_h}$. But $g_c = g_C - g_A - g_L$ since $c(t) = C(t)/A(t)L(t)$. This implies

$$\begin{aligned} g_C &= g_A + g_h + g_L \\ &= g_K \end{aligned}$$

The production function also implies

$$g_Y = g_K = g_C = g_A + g_h + g_L.$$

Hence,

$$\begin{aligned} (RHS) &= \rho + \sigma(g_A + g_h) \\ &= \rho + \sigma(g_C - g_L) \end{aligned}$$

- Therefore, the eq. finally becomes

$$\begin{aligned} r(t) - \delta_K &= \frac{\alpha}{K(t)/Y(t)} - \delta_K \\ &= \rho + \sigma(g_C - g_L) \end{aligned}$$

This is Euler equation! We obtained it before. It is obvious that r and K/Y ratio are constant, so

$$\begin{aligned} (3''')\ r - \delta_K &= \frac{\alpha}{K/Y} - \delta_K \\ &= \rho + \sigma(g_C - g_L) \end{aligned}$$

- The first equality is the definition of r. (Simply, $r =$MPK.)
- The second equality is the Euler eq.
- So we did all bunch of computation to reach what we already know.
- But we can attempt similar computation for other eqs.
- **Part 2/4**: Now (4'):

$$\begin{aligned} (4')\ B - (\delta_h + g_L) &= \eta + \sigma g_c \\ &= \rho - g_A(1-\sigma) - g_L + \sigma(g_C - g_A - g_L) \end{aligned}$$

So

$$\begin{aligned} B - \delta_h &= \rho - g_A(1-\sigma) + \sigma(g_C - g_A - g_L) \\ &= \rho - g_A + \sigma(g_C - g_L) \end{aligned}$$

So

$$B + g_A - \delta_h = \rho + \sigma(g_C - g_L)$$

- Write it with (3"'):

$$(3"')\ \boxed{r-\delta_K} = \boxed{\frac{\alpha}{K/Y}-\delta_K}$$
$$= \boxed{\rho+\sigma(g_C-g_L)}$$
$$= \boxed{B+g_A-\delta_h}$$

- Note that B is MP of human capital in human capital production! (Recall (human capital production)= $Bu(t)h(t)$.)

- So it says

$$\boxed{\text{(net interest rate)}} = \boxed{\text{(MPK in Y production)} - \delta_K}$$
$$= \boxed{\text{(a function of consumption growth)}}$$
$$= \boxed{\text{(MPH in H production)} + g_A - \delta_h}$$

- Interestingly, g_A is in MPH only, not in MPK.

- We can also manipulate the FOCs to relate (MPH in Y production) to here. I will not do this here. Maybe at the problem set.

- **Part 3/4**: Now (A'):

$$(\text{A'})\ \left[(1-u)\frac{h(t)}{k(t)}\right]^{1-\alpha} - \frac{c(t)}{k(t)} - (\delta_K+g_A+g_L) = g_k$$
$$\left[\frac{A(t)(1-u)h(t)L(t)}{K(t)}\right]^{1-\alpha} - \frac{C(t)}{K(t)} - (\delta_K+g_A+g_L) = g_C - g_A - g_L$$
$$\frac{Y(t)}{K(t)} - \frac{C(t)}{Y(t)}\frac{Y(t)}{K(t)} - \delta_K = g_C$$
$$\left(1-\frac{C(t)}{Y(t)}\right)\frac{Y(t)}{K(t)} - \delta_K = g_C$$
$$\frac{s}{K/Y} = g_C + \delta_K$$

Same as before! This is the law of motion of K.

- **Part 4/4**: Now (A'):

$$(\text{B'})\ g_h = Bu - (\delta_h + g_L)$$
$$g_Y - g_A + \delta_h = Bu$$

- This is about the law of motion of H.
- **F. Calibration**
- All equations:

$$\frac{s}{K/Y} = g_C + \delta_K \text{ (same as before)}$$
$$r - \delta_K = \frac{\alpha}{K/Y} - \delta_K \text{ (same as before)}$$
$$= \rho + \sigma(g_C - g_L) \text{ (same as before)}$$
$$= B + g_A - \delta_h \text{ (new)}$$
$$g_Y - g_A + \delta_h = Bu \text{ (new)}$$

- Let's use:
- $s = 0.20$ (same as before)
- $g_L = 0.011$ (same as before)
- $g_Y = g_C = 0.033$ (same as before)
- $\delta_K(K/Y) = 0.12$ (same as before)
- $\alpha = 0.32$ (same as before)
- In addition:
- $\delta_h = 0.033$

  individual depreciation (forgetting) is about 0.013 (Arrazola and de Hevia (2004), ...)

  2.2% of the labor force retires every year, assuming a worker's working lifetime is 45 years

  (Can we improve?)
- $u = 0.28$

  (a) The labor income in educational services and scientific R&D services is 6.7% of GDP

  (b) The foregone labor income due to schooling is worth 3.7% of GDP

  (c) The foregone labor income due to on-the-job training is worth 5.9% of GDP

(d) The foregone labor income of parents, due to in-home training before formal schooling, is worth 6.9% of GDP

The sum of these four is 23.2% of GDP.

On the other hand, the labor income outside the sectors of educational services and scientific R&D services is 61.1% of GDP.

Therefore, comparing these two values, 23.2% and 61.1%, gives $u = 0.28$.

- So

$$\begin{aligned}\frac{0.20}{K/Y} &= 0.033 + \delta_K \text{ with } \delta_K K/Y = 0.12\\ r - \delta_K &= \frac{0.32}{K/Y} - \delta_K \text{ (same as before)}\\ &= \rho + \sigma(0.033 - 0.011) \text{ (same as before)}\\ &= B + g_A - 0.033 \text{ (new)}\\ 0.033 - g_A + 0.033 &= 0.28B \text{ (new)}\end{aligned}$$

- The first 3 eqs. are exactly the same! We have

$$K/Y = 2.42,\ \delta_K = 0.05,\ r = 13.2\%$$
$$\text{If } \sigma = 1, \text{ then } \rho = 0.06$$

- The last 2 eqs. have 2 unknowns, B and g_A.
- Eliminate g_A to have:

$$0.033 + (B - 0.033 - 0.132 + 0.050) + 0.033 = 0.28B.$$
$$\text{So } B = 0.068$$

Then,

$$g_A = 0.047$$

- What??? This can't be right!
- We know

$$\underbrace{g_Y - g_L}_{2.2\%} = \underbrace{g_A}_{4.7\%} + \underbrace{g_h}_{??????}$$

- What might be wrong?
- (A) Physical input in H production? It is OK.

- (B) B is also increasing. (Technology for H production is improving?)
- (C) Human capital externalities?
- We will stop here.
- How to clearly distinguish g_h from g_A?
- See Rangazas (2005, ...) for a recent treatment.

3. Human Capital Externalities

- Static Externalities: Lucas (1988)

$$Y(t) = K(t)^{\alpha}(A(t)(1-u(t))h(t)L(t))^{1-\alpha} \underbrace{\overline{h}(t)^{\gamma}}_{\substack{\text{static} \\ \text{externalities}}},$$

where

$h(t)$: his/her own human capital stock,

$\overline{h}(t)$: average human capital stock in this economy, taken as exogenous by individual consumer

- How large is γ? No consensus.
- Micro study: Mincer regression with additional explanatory variable (average educational attainment of the state, etc.).

  - log $(\text{wage})_i = \alpha + \beta_1(\text{age})_i + \beta_2(\text{education})_i + \beta_3(\text{education of co-workers})_i + ...$

  - Acemoglu and Angrist (2000), Ciccone and Peri (2006): β_3 is small, perhaps negligible.

  - Moretti (2004), Liu (2008): β_3 is large.

- Learning Externalities (Tamura (1991)): Human capital accumulation is now

$$\dot{h}(t) = B\left[u(t)h(t)\right]^{1-\theta} \underbrace{\overline{h}(t)^{\theta}}_{\substack{\text{learning} \\ \text{externalities}}} - (\delta_h + g_L)h(t).$$

- How large is θ? No consensus.

- Micro study: Borjas (1992, 1995): Large.
- Macro study: Choi (2011): $\theta = 0.44$ (large).
- Why does the government subsidize education?
- (1) Externalities
- (2) Imperfect capital market
- (3) ?
- So what is the optimal level of subsidies on education and R&D?: Open question.

4. Review: Economic Growth
Sources of Economic Growth

- Contribution from K accumulation: small or none on the BGP
- So what is the remaining source? It is about the "**ideas**" (knowledge, technologies, know-how, ...).
  - Problem: Not measurable
  - Contribution from **education**: some, but we are not sure about how to measure it
  - Contribution from a decline in **capital goods prices**: some, but what is it ultimately?
  - R&D?
  - ...

——————- up to here: U.S.
——————- from here: XS of economies

- Ideas flow internationally.
- Again, Problem: Not measurable. How do they flow?
  - Mechanical models assume that followers' productivities depend on the leaders'.
  - But why? Through international trade? Exactly how?
  - **Need**: A growth model attached with **international trade**, that matches with stylized facts. This should explain the long-run growth rates of all countries.

EXERCISES

1. True or False? Explain: Each individual will determine his or her optimal level of schooling. There is no need for the government to intervene in education.

   Answer: *FALSE. The government may want to subsidize for the following two reasons. (i) The capital market is not perfect, so young people may not be able to borrow enough to attend the school. (ii) There are human capital externalities. Workers benefit from the human capital stocks owned by other workers. In individual decision of schooling, this social benefit is disregarded, so education is under-invested, which calls for a government intervention.*

2. Pick up any three country-level economies. For each economy, discuss how human capital accumulation (through education) had been quantitatively important in its per-capita output growth. Give a decomposition, for example, as follows:

$$\underbrace{\text{(Per-capita Output Growth)}}_{\substack{2.5\% \text{ per year} \\ (100\%)}} = \underbrace{\text{(Contribution from H Accumulation)}}_{\substack{1.0\% \text{ per year} \\ (40\%)}} + \underbrace{\text{(Misc.)}}_{\substack{1.5\% \text{ per year} \\ (60\%)}}$$

   Measure per-capita human capital stock at time t as $h(t) = \exp(0.1 \times S(t))$, where $S(t)$ is the average years of schooling for workers of 25 years and above. Use the start year (e.g., 1960) and the end year (e.g., 2000) that are common to your three economies. Determine those years based on data availability.

   - The data on years of schooling can be found at Barro, Robert J. and Jong-Wha Lee (2000), "International Data on Educational Attainment: Updates and Implications," CID Working Paper 42. Visit http://www.cid.harvard.edu/ciddata/ciddata.html and click "Appendix Data Tables".
   - The data on per-capita output growth can be found at many places, but Angus Maddison provides a comprehensive dataset. Click "Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD" at http://www.ggdc.net/maddison. Make sure you use "per-capita GDP" instead of "(total) GDP".