Asset Pricing Lecture Notes

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Contents

Fundamental Equation of Asset Pricing

References

- Cochrane, J. H., (2005), Asset Pricing, Revised Edition, Princeton University Press, Princeton, NJ, Chapter 1.
- Ljungqvist, L., and T. J. Sargent (2004), Recursive Macroeconomic Theory, 2nd Edition, MIT Press, Cambridge, MA, Chapters 7-8.

- **Main Question**:

  How are the prices of assets determined?

  ——— -> govt securities, stocks, derivatives, housing, human capital, all others

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1. Fundamental Equation

- Representative consumer/investor
- Endowment economy: Y_0, Y_1, Y_2, ..., Y_t, ... (in units of physical goods)

  (stochastic)

- No storage.
- But allowed to buy or sell a specific **one-period asset**
- This one-period asset is between periods 0 and 1. It can be one of any assets you can imagine (stocks, bonds, human capital, ...)

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  Period 0 Period 1

  Pay P_0 Receive X_1
- P_0: price of the asset at period 0 (in units of physical goods)
- X_1: (stochastic) payoff at period 1 (in units of physical goods)
- The consumer observes at period 0: Y_0, P_0, distribution of Y_t's for $t = 1, 2, ...$, distribution of X_1, any stochastic properties of Y_t's and X_1
- The consumer does not observe at period 0: **Realized** values of X_1, Y_1, Y_2, ...
- The consumer purchases a units of this asset at period 0.

  (If $a < 0$, she sells.)
- The consumer maximizes at period 0:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right], \quad 0 < \beta < 1.$$

  Here, as usual, $u' > 0$ and $u'' < 0$.
- Subscript 0 emphasizes that she is at period 0. (That is, E is based on **all relevant information available** at period 0.)
- Constraints:

$$C_0 = Y_0 - aP_0,$$
$$C_1 = Y_1 + aX_1,$$
$$C_2 = Y_2, ...$$

- Note: This is an endowment economy. We know $a = 0$ so that $C_0 = Y_0$ and $C_1 = Y_1$. Then why are we solving it? We want to find P_0 which **clears** the market.

- Unconstrained maximization problem:

$$\max_a E_0\left[u(Y_0 - aP_0) + \beta u(Y_1 + aX_1) + \beta^2 u(Y_2) + ...\right]$$
$$\text{So } \max_a u(Y_0 - aP_0) + \beta E_0\left[u(Y_1 + aX_1)\right] + \beta^2 E_0\left[u(Y_2)\right] + ...$$

- Under some regularity condition (that enable us to differentiate in the expectation), FOC:

$$-P_0 u'(\underbrace{Y_0 - aP_0}_{=C_0}) + \beta E_0\left[X_1 u'(\underbrace{Y_1 + aX_1}_{C_1})\right] = 0.$$

- Hence,

$$\boxed{P_0 = E_0\left[\tfrac{\beta U'(C_1)}{U'(C_0)} X_1\right]}.$$

This is called the **fundamental equation of asset pricing** or **Euler equation**.

- Now write it more generally: Period 0 —–> Period t, and Period 1 —–> Period $t+1$.

- This equation prices **any** given asset j.

- Fundamental eq.:

$$\underbrace{P_t^j}_{\text{Price of Asset } j \text{ at Period } t} = E_t[m_{t+1} \underbrace{X_{t+1}^j}_{\text{Stochastic Payoff of Asset } j \text{ at Period } t+1}] \quad .....(1)$$

where

$$m_{t+1} \equiv \frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{\beta u'(Y_{t+1})}{u'(Y_t)}.$$

- m_{t+1} is called the **stochastic discount factor**, the **pricing kernel**, or the **marginal rate of substitution**. It captures the **macroeconomic risk** (or **endowment risk**, or **consumption risk**).

- **Example**: $u(C) = C^{1-\sigma}/(1-\sigma)$, $\sigma > 0$, $\sigma \neq 1$ (CRRA) and $u(C) = \log C$ when $\sigma \longrightarrow 1$. Then

$$u'(C) = C^{-\sigma}.$$

Hence,

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} = \beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}.$$

- We may also write the fundamental eq. as

$$1 = E_t\left[m_{t+1}\frac{X^j_{t+1}}{P^j_t}\right]$$
$$= E_t\left[m_{t+1}(1+r^j_{t+1})\right] \quad .....(2)$$

- r^j_{t+1}: (net) **rate of return**, or **return**, on asset j between t and $t+1$.

- **Example**: Consider a **risk-free asset** that pays 1 unit of physical good at $t+1$. From (1),

$$P^f_t = E_t[m_{t+1}] = E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right].$$

The gross return on this asset, or the gross **risk-free rate**, is

$$R^f_t \equiv 1 + \underbrace{r^f_t}_{\text{Net Risk-free Return between } t \text{ and } t+1} = \frac{1}{P^f_t} = \frac{1}{E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right]}.$$

- Notice that the subscripts in R^f_t and r^f_t is t, even though this asset pays at $t+1$. This is because this asset is risk-free. The realized payoffs and returns at period $t+1$ are known in advance at period t.

- This formular can be also obtained directly from (2):

$$1 = E_t[m_{t+1}(1+r^f_t)] = (1+r^f_t)E_t[m_{t+1}].$$

2. Comovement of Consumptions and Payoffs

- (1):

$$P^j_t = E_t[m_{t+1}X^j_{t+1}]$$
$$= E_t[m_{t+1}]E_t[X^j_{t+1}] + cov_t[m_{t+1}, X^j_{t+1}]$$

- In the Example above,

$$E_t[m_{t+1}] = \frac{1}{1+r^f_t}$$

- Hence,

$$P^j_t = \frac{E_t[X^j_{t+1}]}{1+r^f_t} + cov_t[m_{t+1}, X^j_{t+1}]$$

- **Decomposition of Asset Price**:
- (i) First Term: Expected value of future payment, discounted by the risk-free rate (Expected value $\uparrow \Longrightarrow$ Price $\uparrow$)
- (ii) Second Term: Under CRRA,

$$\begin{aligned} cov_t[m_{t+1}, X^j_{t+1}] &= cov_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}, X^j_{t+1}\right] \\ &= cov_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, X^j_{t+1}\right] \\ &= \underbrace{corr_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, X^j_{t+1}\right]}_{\text{(a)}}\underbrace{sd_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right]}_{\text{(b)}}\underbrace{sd_t\left[X^j_{t+1}\right]}_{\text{(c)}} \end{aligned}$$

  where $corr_t$ is the correlation and sd_t is the standard deviation.

- (a): So if $corr_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, X^j_{t+1}\right] \downarrow \Longrightarrow$ Price $\downarrow$ and vice versa
  - **Economic interpretation**: If $\frac{Y_{t+1}}{Y_t}$ and X^j_{t+1} tend to "co-vary" more closely (i.e., more strongly positively correlated)
    $\Longrightarrow$ (If this asset pays more when the economy is good)
    $\Longrightarrow$ $corr_t\left[\frac{Y_{t+1}}{Y_t}, X^j_{t+1}\right]$ is high
    $\Longrightarrow$ $corr_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, X^j_{t+1}\right]$ is low
  - The risk-averse consumer does not prefer an asset paying higher payoffs when the economy is already good. Hence, this asset is less attractive. This is a cheap asset!
  - The assets that pay more in recession are more attractive. The consumer wants to hold them for **insurance**. So they are more expensive.
  - (a) reflects how payoffs are related to m_{t+1}. In other words, the second term reflects the **macroeconomic risk** (or **endowment risk**, or **consumption risk**.
- (b) is common to all assets, so I will skip. (Still, it can be an interesting term!)
- (c) is the **volaility** of this asset's payoffs.

- To summarize, an asset price is determined by

  1. Expected future payoff (First Term)
  2. Correlation between future payoff and macroeconomic shock, in this model, the consumption growth (Part (a) of the Second Term)
  3. Volatility of future payoff (Part (c) of the Second Term)

- Another way to see the same implication (this time with r):

$$\begin{aligned} 1 &= E_t[m_{t+1}(1+r^j_{t+1})] \\ &= E_t[m_{t+1}]E_t[1+r^j_{t+1}] + cov_t[m_{t+1}, 1+r^j_{t+1}] \\ &= \frac{E_t[1+r^j_{t+1}]}{1+r^f_t} + cov_t[m_{t+1}, 1+r^j_{t+1}] \end{aligned}$$

- So

$$E_t[1+r^j_{t+1}] = 1 + r^f_t - (1+r^f_t)cov_t[m_{t+1}, 1+r^j_{t+1}]$$

  So

$$\begin{aligned} E_t[r^j_{t+1}] &= r^f_t - \underbrace{(1+r^f_t)cov_t[m_{t+1}, 1+r^j_{t+1}]}_{\text{"risk adjustment"}} \\ &= r^f_t - (1+r^f_t)cov_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, 1+r^j_{t+1}\right] \end{aligned}$$

- **Economic Interpretation**: If $\frac{Y_{t+1}}{Y_t}$ and r^j_{t+1} tend to "co-vary"

$\Longrightarrow$ (If this asset has higher return when the economy is good)

$\Longrightarrow$ $corr_t\left[\frac{Y_{t+1}}{Y_t}, 1+r^j_{t+1}\right]$ is high

$\Longrightarrow$ $corr_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, 1+r^j_{t+1}\right]$ is low

$\Longrightarrow$ $E_t[r^j_{t+1}]$ is high

$\Longrightarrow$ (Expected return should be high!)

$\Longrightarrow$ "This asset pays more when the economy is already good, so I don't want it. If you want me to hold it, pay me more."

3. ER-β Representation

- (2) again:

$$\begin{aligned} 1 &= E_t[m_{t+1}(1+r_{t+1}^j)] \\ &= E_t[m_{t+1}]E_t[1+r_{t+1}^j] + cov_t[m_{t+1}, 1+r_{t+1}^j] \end{aligned}$$

- So

$$\begin{aligned} E_t[1+r_{t+1}^j] &= \frac{1}{E_t[m_{t+1}]} - \frac{cov_t[m_{t+1}, 1+r_{t+1}^j]}{E_t[m_{t+1}]} \\ &= 1 + r_t^f - \frac{cov_t[m_{t+1}, 1+r_{t+1}^j]}{E_t[m_{t+1}]} \end{aligned}$$

- So

$$E_t[r_{t+1}^j] = r_t^f + \underbrace{\frac{cov_t[m_{t+1}, 1+r_{t+1}^j]}{var_t[m_{t+1}]}}_{\text{(A)}} \underbrace{\left(-\frac{var_t[m_{t+1}]}{E_t[m_{t+1}]}\right)}_{\text{(B)}}$$

- (A): $\frac{cov(X,Y)}{var(X)}$ is a slope of a regression of Y on X. Define

$$\text{(A)} \quad \frac{cov_t[m_{t+1}, 1+r_{t+1}^j]}{var_t[m_{t+1}]} \equiv \beta_t^j.$$

- β_t^j: "When $m_{t+1} = \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}$ increases by 1, by how much $1 + r_{t+1}^j$ increases?"

- Again, $m_{t+1} = \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}$ reflects the **risk** of this economy.

- β_t^j measures how sensitive the asset is to the risk. That is, β_t^j measures the "quantity" of the macroeconomic risk that an investor buys when he purchases this asset.

- β_t^j is often called the **quantity of risk** or simply **beta**.

- (B): Same for all assets.

$$\text{(B)} \quad -\frac{var_t[m_{t+1}]}{E_t[m_{t+1}]} \equiv \lambda_t$$

- λ_t is often called the **price of risk**, common to all assets.

- **ER-β representation**:

$$E_t[r_{t+1}^j] = r_t^f + \beta_t^j \lambda_t.$$

- So λ_t is given in macroeconomy. Each asset is different becasue β_t^j is different.

4. CAPM as a Special Case

- Assume $\sigma = 1$ (log utility).
- Previously: We analyzed a one-period asset.
- Consider the **wealth portfolio (market portfolio)**: Pay P_t^W at t. Receive Y_{t+1}, Y_{t+2}, ... later.
- Set-Up:————+————————————————-+-

  t t+1

  At the beginning of each period: Dividends paid

  At the end of each period: Assets traded

- Now:

$$\max_{a^W} E_t\left[\sum_{\tau=0}^{\infty}\beta^\tau \log(C_{t+\tau})\right]$$

s.t.

$$\begin{aligned} C_t &= Y_t - a^W P_t^W, \\ C_{t+1} &= Y_{t+1} + a^W Y_{t+1} = (1+a^W)Y_{t+1}, \\ C_{t+2} &= (1+a^W)Y_{t+2}, ... \end{aligned}$$

- Unconstrained problem:

$$\max_{a^W} \log(Y_t - a^W P_t^W) + E_t\left[\sum_{\tau=1}^{\infty}\beta^\tau \log((1+a^W)Y_{t+\tau})\right]$$

- FOC:

$$-P_t^W \frac{1}{Y_t - a^W P_t^W} + E_t\left[\sum_{\tau=1}^{\infty}\beta^\tau \frac{1}{(1+a^W)Y_{t+\tau}} Y_{t+\tau}\right] = 0.$$

- In equil, $a^W = 0$. Hence,

$$-\frac{P_t^W}{Y_t} + \sum_{\tau=1}^{\infty}\beta^\tau = 0.$$

Hence,

$$P_t^W = Y_t \sum_{\tau=1}^{\infty}\beta^\tau = Y_t \frac{\beta}{1-\beta}$$

using $\sum_{\tau=0}^{\infty}\beta^\tau = 1/(1-\beta)$ for $0 < \beta < 1$.

- So

$$1 + r^W_{t+1} \equiv \frac{Y_{t+1} + P^W_{t+1}}{P^W_t} = \frac{Y_{t+1} + Y_{t+1}\frac{\beta}{1-\beta}}{Y_t\frac{\beta}{1-\beta}} = \frac{1}{\beta}\frac{Y_{t+1}}{Y_t}.$$

- Wait! This is the stochastic discount factor! That is,

$$m_{t+1} = \frac{\beta u'(Y_{t+1})}{u'(Y_t)} = \frac{\beta}{Y_{t+1}/Y_t} = \frac{1}{1 + r^W_{t+1}}$$

- **Conclusion**: By assuming log utility, the stochastic discount factor, m_{t+1}, can be replaced by a function of r^W_{t+1}! So for any asset j,

$$(2)\ 1 = E_t[m_{t+1}(1 + r^j_{t+1})] = E_t\left[\frac{1 + r^j_{t+1}}{1 + r^W_{t+1}}\right]$$

- Now we don't need to consider the consumption growth or endowment growth.
- This representation is called **CAPM** (**capital asset pricing model**).
- After some approximation, the CAPM is often written as

$$E_t[1 + r^j_{t+1}] = \alpha^j_t + \underbrace{\beta^j_t}_{\text{CAPM beta}} E_t[1 + r^W_{t+1}]$$

5. Arrow-Debreu Market Revisited

- Now a new set-up. This is another way to understand the fundamental eq.
- Today is period 0.
- At time 0, consumers trade "claims" to future physical goods for all possible future "histories."
- Example: $s_0 = 1$ ("good") ————> $s_1 = 1$ ("good")

  $s_1 = 2$ ("bad")

  $s_0 = 2$ ("bad") ————> $s_1 = 1$ ("good")

  $s_1 = 2$ ("bad")

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  History: set of realized states

- $q_t^0(s^t)$: price at period 0, in units of time-0 physical goods, of an asset (or a security) that pays 1 unit of time-t physical good if history $s^t \equiv (s_0, s_1, ...)$ occurs, and 0 otherwise.

- Example:

  - Suppose $s_0 = 1$ (today's state is #1).
  - $q_0^0(\underbrace{s_0 = 1}_{\text{known}}) = 1$ (The price of today's physical good is, of course, 1 unit of today's physical good.)
  - $q_1^0(\underbrace{s_0 = 1}_{\text{known}}, \underbrace{s_1 = 1}_{\text{possible future scenario}})$: Price of a security paying 1 at time 1 if $s_1 = 1$ and 0 otherwise
  - $q_1^0(\underbrace{s_0 = 1}_{\text{known}}, \underbrace{s_1 = 1, s_2 = 2}_{\text{possible future scenario}})$: ...

- The problem of consumer i is to maximize

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t^i) \right], 0 < \beta < 1,$$

or,

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) prob(s^t | s_0)$$

- BC: You sell all your claims to future **contingent endowments**. Then you buy all future **contingent consumptions**.

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \underbrace{\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \underbrace{y_t^i(s^t)}_{\text{time-}t\text{ endowment contingent on } s^t}}_{\text{value today of all } i\text{'s future contingent endowments}}$$

- Lagrangian:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) prob(s^t | s_0) + \lambda^i \left[\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) - \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \right]$$

- FOC wrt $c_\tau^i(s^\tau)$:

$$\beta^t u'(c_\tau^i(s^\tau)) prob(s^\tau | s_0) = \lambda^i q_\tau^0(s^\tau)$$

for all j, τ and s^t.

- Hence,

$$q^0_\tau(s^\tau) = \frac{\beta^\tau u'(c^i_\tau(s^\tau)) prob(s^\tau|s_0)}{\lambda^i}$$
$$1 = \frac{u'(c^i_0(s^0))}{\lambda^i}$$

Hence,

$$q^0_\tau(s^\tau) = \beta^\tau \frac{u'(c^i_\tau(s^\tau))}{u'(c^i_0(s^0))} prob(s^\tau|s_0)$$

- Looks familiar? Related to the Fundamental eq. for one-period asset:

$$(1)\ P_t = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} X_{t+1}\right]$$

- In a **competitive equilibrium** of this Arrow-Debreu economy,
- (i) Each consumer i solves the problem.
- (ii) The solution $\{c^i_t(s^t)\}$ is feasible, i.e.,

$$\sum_i c^i_t(s^t) \leq \sum_i y^i_t(s^t), \text{ for all } t \text{ and } s^t.$$

- Any asset can be broken into a set of Arrow-Debreu securities.
- **Example**: The price of a risk-free security that pays 1 in period 1:

$$\sum_{s_1} q^0_1(s_0, s_1)$$

- **Example**: The price of a security that pays $d(s_t)$ when period-t state is s_t is

$$\sum_{t=0}^{\infty} \sum_{s^t} q^0_t(s^t) d(s_t)$$

EXERCISES

1. True or False? Explain: A share price of Microsoft is an expected value of the sum of "its future dividends discounted by risk-free rates." That is, assuming for simplicity that Microsoft delivers dividends only at period $t+1$, its share price at t, denoted by P_t^{Micro}, is $P_t^{Micro} = E_t[X_{t+1}^{Micro}]/(1+r_t^f)$, where X_{t+1}^{Micro} is the stochastic dividend delivered at $t+1$, and r_t^f is the risk-free rate between t and $t+1$.

   Answer: *FALSE. We have $P_t^i = \frac{E_t[X_{t+1}^i]}{1+r_t^f} + cov_t[m_{t+1}, X_{t+1}^i]$. The given statement is only about the first term. There is a second term which is about the co-movement of macroeconomic risk and the payoffs.*

2. True or False? Explain: Theoretically, the risk-free rate should be constant over time. That is, $r_1^f = r_2^f = ... = r_t^f = ...$ where r_t^f is the risk-free rate at period t.

   Answer: *False. The fundamental equation implies*

$$E_t[m_{t+1}] = \frac{1}{1+r_t^f}$$

   where m_{t+1} is a stochastic discount factor at $t+1$. But $E_t[m_{t+1}]$ moves over time, and hence, r_t^f also moves over time. To see this, consider

$$E_t[m_{t+1}] = E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right]$$

   under CRRA utility. This changes as today's (period t's) anticipation on tomorrow's (period (t+1)'s) economic growth changes. In other words, the risk-free rate depends on the anticipation about the future economy.

3. True or False? Explain: Theoretically, an asset whose return has a higher standard deviation should always have a higher expected return.

   Answer: *The fundamental equation of asset pricing can be written as*

$$\begin{aligned} 1 &= E_t[m_{t+1}(1+r_{t+1}^j)] \\ &= E_t[m_{t+1}]E_t[1+r_{t+1}^j] + cov_t[m_{t+1}, 1+r_{t+1}^j] \\ &= \frac{E_t[1+r_{t+1}^j]}{1+r_t^f} + cov_t[m_{t+1}, 1+r_{t+1}^j] \end{aligned}$$

where m_{t+1} is a stochastic discount factor at $t+1$, r^j_{t+1} is the return on asset j at $t+1$, and r^f_t is the risk-free rate between t and $t+1$. Then,

$$E_t[1+r^j_{t+1}] = 1 + r^f_t - (1+r^f_t)cov_t[m_{t+1}, 1+r^j_{t+1}]$$

So

$$\begin{aligned} E_t[r^j_{t+1}] &= r^f_t - (1+r^f_t)cov_t[m_{t+1}, 1+r^j_{t+1}] \\ &= r^f_t - (1+r^f_t)cov_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, 1+r^j_{t+1}\right] \\ &= r^f_t - (1+r^f_t)corr_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, 1+r^j_{t+1}\right] \\ &\times sd_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right] \times sd_t\left[1+r^j_{t+1}\right] \end{aligned}$$

This implies that there is a negative relationship between $E_t[r^j_{t+1}]$ and $sd_t\left[1+r^j_{t+1}\right] = sd_t\left[r^j_{t+1}\right]$ when $corr_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}, 1+r^j_{t+1}\right]$ is controlled for. The statement is not always true.

4. True or False? Explain: Theoretically, there can be no asset that provides an expected rate of return lower than the risk-free rate. That is, there can be no asset j such that $E_t[r^j_{t+1}] < r^f_t$, where r^j_{t+1} is asset j's rate of return between t and $t+1$ and r^f_t is a risk-free rate between t and $t+1$.

   Answer: *FALSE. The fundamental equation suggests*

$$E_t[r^j_{t+1}] = r^f_t - \underbrace{(1+r^f_t)cov_t[m_{t+1}, 1+r^j_{t+1}]}_{\text{"risk adjustment"}}.$$

   If the cov_t part is negative (so asset j tends to deliver more units of goods in recession), then $E_t[r^j_{t+1}]$ is higher than r^f_t.

5. The fundamental equation of asset pricing is $P^j_t = E_t[m_{t+1}X^j_{t+1}]$, where P^j_t is the price of one-period asset j at period t, m_{t+1} is the stochastic discount factor between t and $t+1$ for any one-period asset, and X^j_{t+1} is a stochastic payoff of asset j at period $t+1$. Obtain m_{t+1} for the following preferences, which are often called the "constant absolute risk aversion" (CARA):

$$E_t\left[\sum_{\tau=0}^{\infty}\beta^{\tau}\frac{1}{\alpha}\exp\left(-\alpha C_{t+\tau}\right)\right].$$

Answer: *Here,* $u(C) = \frac{1}{\alpha}\exp(-\alpha C)$, *so* $u'(C) = -\exp(-\alpha C)$. *Hence,* $m_{t+1} = \beta\frac{u'(C_{t+1})}{u'(C_t)} = \beta\frac{-\exp(-\alpha C_{t+1})}{-\exp(-\alpha C_t)} = \beta\frac{\exp(-\alpha C_{t+1})}{\exp(-\alpha C_t)}$.

6. There are two periods, 0 and 1. There are two possible states, "peace" and "war". We are at the beginning of period 0, and the state at period 0 is *known* to be "peace". The state at period 1 will be "peace" with probability π and "war" with probability $1-\pi$, respectively.

   The endowment to this economy at each period is 1 unit of physical good, *regardless of the state*. The government consumption at each period, which is given exogenously, is g_P units of physical goods when the state is "peace", or g_W units when the state is "war", where $g_W > g_P$. (In other words, it spends more in "war.") The government consumptions should be financed by distortionary taxes, but the government also has an access to the Arrow-Debreu market. The government can purchase or sell *any* amounts of claims to period-1 physical goods contingent on period-1 state, at prices given by

$$q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"peace"}) = \frac{\pi}{1+r}, \tag{1}$$

$$q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"war"}) = \frac{1-\pi}{1+r}, \tag{2}$$

   where $r > 0$ is a constant and can be interpreted as a risk-free interest rate. For example, if the government purchases 1 claim to a period-1 physical good contingent on "peace" by paying $\pi/(1+r)$ units at period 0, the government at period 1 will receive 1 unit of physical good if the state is "peace" and 0 unit if the state is "war".

   The government minimizes the present expected value of dead-weight losses:

$$W(\tau_0) + \frac{1}{1+r}\left[\pi W(\tau_{1P}) + (1-\pi)W(\tau_{1W})\right].$$

   Here, W is a twice differentiable function satisfying $W(0) = 0$, $W'(x) > 0$, and $W''(x) > 0$ for $x \geq 0$. Also, τ_0 is the overall tax rate at period 0, and τ_{1P} and τ_{1W} are the overall tax rates at period 1 in states "peace" and "war", respectively. At the beginning of period 0, the government designs the optimal fiscal policy, by determining the period-0 overall tax rate (τ_0) and by providing its "plan" on the period-1 overall tax rate contingent on the state (τ_{1P} and τ_{1W}). Notice that at period 1, the government is allowed to impose different overall tax rates depending on the state.

   Go as far as you can to characterize the optimal fiscal policy. Your answer should be sufficient to answer the following questions:

   (a) At period 1, when should the overall tax rate be higher, in "peace" or in "war"? (That is, compare the levels of τ_{1P} and τ_{1W} chosen by the government.)

(b) Recall that the period-0 state is known to be "peace". If the period-1 state also happens to be "peace", then when should the overall tax rate be higher, in period 0 or in period 1, while the two periods have the same states? (That is, compare the levels of τ_0 and τ_{1P} chosen by the government.)

Note: We have assumed that the Arrow-Debreu prices are exogenously given as (1) and (2). This is reasonable. (i) They satisfy

$$q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"peace"}) + q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"war"}) = \frac{1}{1+r}$$

so the price for risk-free asset is correct. (ii) And those prices are proportional to probabilities.

Answer: *The government's problem is to choose τ_0, τ_{1P} and τ_{1W} to minimize the objective function (given in the question) subject to*

$$\begin{aligned} g_P + q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"peace"}) \times g_P + q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"war"}) \times g_W \\ \leq \tau_0 + q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"peace"}) \times \tau_{1P} + q_1^0(s_0 = \text{"peace"},\ s_1 = \text{"war"}) \times \tau_{1W}. \end{aligned}$$

Using (1) and (2), the Lagrangian function is

$$\begin{aligned} L = {} & W(\tau_0) + \frac{1}{1+r}\left[\pi W(\tau_{1P}) + (1-\pi)W(\tau_{1W})\right] \\ & - \lambda\left[\tau_0 + \frac{\pi}{1+r}\tau_{1P} + \frac{1-\pi}{1+r} \times \tau_{1W} - g_P - \frac{\pi}{1+r}g_P - \frac{1-\pi}{1+r} \times g_W\right] \end{aligned}$$

The first-order conditions are

$$\begin{aligned} W'(\tau_0) &= \lambda, \\ W'(\tau_{1P}) &= \lambda, \\ W'(\tau_{1W}) &= \lambda. \end{aligned}$$

So

$$\tau_0^* = \tau_{1P}^* = \tau_{1W}^*.$$

7. There are 2 periods (periods 0 and 1), 2 states (states #1 and #2), and 2 consumers (consumers A and B). The two consumers are at the end of period 0, maximizing the expected utility of period 1. To be specific, consumer A maximizes $\frac{2}{3}\log(c_1^A) + \frac{1}{3}\log(c_2^A)$, assigning $\frac{2}{3}$ and $\frac{1}{3}$ to the probabilities of the occurrences of states #1 and #2, respectively, where c_j^i is consumer i's consumption at state #j. Symmetrically, Consumer B maximizes $\frac{1}{3}\log(c_1^B) + \frac{2}{3}\log(c_2^B)$, assigning $\frac{1}{3}$ and $\frac{2}{3}$, respectively. The endowments of consumers are as follows at both states. Also, the payoffs of two

assets #1 and #2 are as follows at both states. Of course, the consumers can sell and purchase those assets to and from each other.

Period 1's possible states:	State #1	State #2
Consumer A's endowment	2	1
Consumer B's endowment	1	2
Consumer A's assigned probability	$\frac{2}{3}$	$\frac{1}{3}$
Consumer B's assigned probability	$\frac{1}{3}$	$\frac{2}{3}$
Asset #1's payoff	1	0
Asset #2's payoff	0	1

Solve for an equilibrium and complete the following statement:

"Consumer A's consumption is _____ unit(s) at State #1 and _____ unit(s) at State #2. Consumer B's consumption is _____ unit(s) at State #1 and _____ unit(s) at State #2. Consumer A sells _____ unit(s) of Asset #_____ (1 or 2) to Consumer B. Consumer A purchases _____ unit(s) of Asset #_____ (1 or 2) from Consumer B."

Answer: *Tricky version: Agent A's problem:*

$$\max_{D,\theta} \frac{2}{3}\log(2-D) + \frac{1}{3}\log(1+\theta).$$

(D: Agent A's delivery to Agent B, if State #1 occurs at period 1. θ: Agent B's delivery to Agent A, if State #2 occurs at period 1.) This problem is subject to the budget constraint:

$$q_1 D = q_2 \theta.$$

Note: This set-up of the problem assumes that Agent A would sell Asset #1. You may ask how we know Agent A sells it and does not buy. Yes, it is possible that she buys Asset #1. Then D would be negative, so there is nothing wrong in the above problem.

This is a symmetric problem, so $q_1 = q_2$. Eventually the problem is

$$\max_{D} \frac{2}{3}\log(2-D) + \frac{1}{3}\log(1+D).$$

FOC:

$$\frac{2}{3}\frac{-1}{2-D} + \frac{1}{3}\frac{1}{1+D} = 0.$$

$$\text{So } 2(1+D) = 2-D$$

$$\text{So } D = 0.$$

"Consumer A's consumption is $\boxed{2}$ *units at State #1 and* $\boxed{1}$ *units at State #2. Consumer B's consumption is* $\boxed{1}$ *units at State #1 and* $\boxed{2}$ *units at State #2. Consumer A sells* $\boxed{0}$ *units of Asset #______ (1 or 2) to Consumer B. Consumer A purchases* $\boxed{0}$ *units of Asset #______ (1 or 2) from Consumer B."*

Formal version: Agent A's problem:

$$\max_{c_1^A, c_2^A} \frac{2}{3}\log(c_1^A) + \frac{1}{3}\log(c_2^A)$$

subject to

$$\begin{aligned} q_1 c_1^A + q_2 c_2^A &\leq q_1 y_1^A + q_2 y_2^A \\ &= 2q_1 + q_2 \end{aligned}$$

Eliminate c_2^A *in the objective function to have an unconstrained problem:*

$$\max_{c_1^A} \frac{2}{3}\log(c_1^A) + \frac{1}{3}\log\underbrace{\left(2\frac{q_1}{q_2} + 1 - \frac{q_1}{q_2}c_1^A\right)}_{=c_2^A}.$$

The FOC is

$$\begin{aligned} \frac{2}{3}\frac{1}{c_1^A} + \frac{1}{3}\frac{1}{c_2^A}\left(-\frac{q_1}{q_2}\right) &= 0. \\ \text{So } \frac{2}{3}\frac{1}{c_1^A} &= \frac{1}{3}\frac{1}{c_2^A}\frac{q_1}{q_2}. \\ \text{So } \frac{c_2^A}{c_1^A} &= \frac{1}{2}\frac{q_1}{q_2}. \end{aligned}$$

Going back to the budget constraint,

$$\begin{aligned} q_1 c_1^A + q_2 c_2^A &= 2q_1 + q_2. \\ \text{So } q_1 c_1^A + q_2 \frac{1}{2}\frac{q_1}{q_2} c_1^A &= 2q_1 + q_2. \\ \text{So } \left(q_1 + \frac{1}{2}q_1\right) c_1^A &= 2q_1 + q_2. \\ \text{So } c_1^A &= \frac{2q_1 + q_2}{\frac{3}{2}q_1}. \end{aligned}$$

Agent B's problem:

$$\max_{c_1^B, c_2^B} \frac{1}{3}\log(c_1^B) + \frac{2}{3}\log(c_2^B)$$

subject to

$$q_1 c_1^B + q_2 c_2^B \leq q_1 y_1^B + q_2 y_2^B$$
$$= q_1 + 2q_2$$

Solving in the same way,

$$\frac{c_1^B}{c_2^B} = \frac{1}{2}\frac{q_1}{q_2}.$$
$$c_2^B = \frac{2q_1 + q_2}{\frac{3}{2}q_1}.$$

Now we consider the feasibility (or resource constraint). We should have

$$c_1^A + c_1^B = y_1^A + y_1^B$$

(which means state #1's aggregate consumption = aggregate endowment), and

$$c_2^A + c_2^B = y_2^A + y_2^B.$$

Hence,

$$\underbrace{c_1^A}_{=\frac{2q_1+q_2}{\frac{3}{2}q_1}} + \underbrace{c_1^B}_{=\frac{1}{2}\frac{q_1}{q_2}\frac{2q_1+q_2}{\frac{3}{2}q_1}} = y_1^A + y_1^B = 3.$$
$$\text{So } \frac{2q_1 + q_2}{\frac{3}{2}q_1} + \frac{1}{2}\frac{q_1}{q_2}\frac{2q_1 + q_2}{\frac{3}{2}q_1} = 3.$$
$$\text{So } 2q_1 + q_2 + \frac{1}{2}\frac{q_1}{q_2}(2q_1 + q_2) = 3\frac{3}{2}q_1.$$
$$\text{So } \left(1 + \frac{1}{2}\frac{q_1}{q_2}\right)(2q_1 + q_2) = 3\frac{3}{2}q_1.$$
$$\text{So } (q_1 + 2q_2)(2q_1 + q_2) = 9q_1 q_2.$$

There is a symmetricity. An obvious solution is $q_1 = q_2$*. Then*

$$c_1^A = \frac{2q_1 + q_2}{\frac{3}{2}q_1} = \frac{2+1}{\frac{3}{2}} = 2.$$
$$c_2^A = 1.$$
$$c_1^B = 1.$$
$$c_2^B = 2.$$

For each consumer at each state, consumption equals endowment. Hence, there is no asset traded.

Fundamental Equation of Asset Pricing: Empirical Issues

References

- Cochrane, J. H., (2005), Asset Pricing, Revised Edition, Princeton University Press, Princeton, NJ, Chapter 1.
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Contents: Predictions of the fundamental equation vs. Data

1. Risk-Free Rate
2. Stock Return
3. Term Structure of (Risk-Free) Interest Rates (in a separate section)
4. ...

1. Risk-Free Rate

- **Data**: "Government Bonds" in EconS 320
- Consider a risk-free asset whose payoff is (riskless) 1 unit at $t+1$. From (1), under CRRA,

$$P_t^f = E_t[m_{t+1}] = E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right].$$

- **Assumption**: $\beta = 0.95$, $\sigma = 2$. You can try other numbers if you want.
- **Data** on GDP growth: Mehra and Prescott (1985)
  - (average of $Y_{t+1}/Y_t - 1$)= 0.018.
  - (standard deviation of $Y_{t+1}/Y_t - 1$)= 0.036
  - (autocorrelation of $Y_{t+1}/Y_t - 1$)= $corr(Y_{t+1}/Y_t - 1, Y_{t+2}/Y_{t+1} - 1) = -0.14$
- What is the simplest possible way to introduce (i) uncertainty (standard deviation) and (ii) negative autocorrelation?
- At least at the beginning, you want to remain simple. The simplest possible way to introduce the uncertainty is to assume that there are two possible "states":

$$\frac{Y_{t+1}}{Y_t} - 1 = \begin{cases} 0.018 + 0.036 = 0.054 \text{ (state 1: "good")} \\ 0.018 - 0.036 = -0.018 \text{ (state 2: "bad")} \end{cases}$$

- To introduce autocorrelation, assume a **Markov chain** for $Y_{t+1}/Y_t - 1$. (Time-(t+1) state depends on time-t state only, and nothing else.)
- **Transition matrix**:

$$P = \begin{pmatrix} \text{prob}_{1\to 1} & \text{prob}_{1\to 2} \\ \text{prob}_{2\to 1} & \text{prob}_{2\to 2} \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \begin{pmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{pmatrix}$$

- Here, $p = 0.43$ has come from the implied autocorrelation of -0.14:

$$corr\left(\frac{Y_{t+1}}{Y_t} - 1, \frac{Y_{t+2}}{Y_{t+1}} - 1\right) = \frac{cov\left(\frac{Y_{t+1}}{Y_t} - 1, \frac{Y_{t+2}}{Y_{t+1}} - 1\right)}{sd(\frac{Y_{t+1}}{Y_t} - 1)sd(\frac{Y_{t+2}}{Y_{t+1}} - 1)}$$

$$= \frac{E\left[\left(\frac{Y_{t+1}}{Y_t} - 1\right)\left(\frac{Y_{t+2}}{Y_{t+1}} - 1\right)\right] - \underbrace{E\left[\frac{Y_{t+1}}{Y_t} - 1\right]E\left[\frac{Y_{t+2}}{Y_{t+1}} - 1\right]}_{=0.018^2}}{\underbrace{sd(\frac{Y_{t+1}}{Y_t} - 1)^2}_{=0.036^2}}$$

Trick to compute $E\left[\left(\frac{Y_{t+1}}{Y_t} - 1\right)\left(\frac{Y_{t+2}}{Y_{t+1}} - 1\right)\right]$:

+——> State 1 [Prob: 0.5*0.43] $\left(\frac{Y_{t+1}}{Y_t} - 1\right)\left(\frac{Y_{t+2}}{Y_{t+1}} - 1\right) = 0.054 \times 0.054$

[50%] State 1 +

+——> State 2 [Prob: 0.5*0.57] $\left(\frac{Y_{t+1}}{Y_t}-1\right)\left(\frac{Y_{t+2}}{Y_{t+1}}-1\right)=0.054\times(-0.018)$

+——> State 1 [Prob: 0.5*0.57] $\left(\frac{Y_{t+1}}{Y_t}-1\right)\left(\frac{Y_{t+2}}{Y_{t+1}}-1\right)=(-0.018)\times 0.054$

[50%] State 2 +

+——> State 2 [Prob: 0.5*0.43] $\left(\frac{Y_{t+1}}{Y_t}-1\right)\left(\frac{Y_{t+2}}{Y_{t+1}}-1\right)=(-0.018)\times(-0.018)$

Hence,

$$E\left[\left(\frac{Y_{t+1}}{Y_t}-1\right)\left(\frac{Y_{t+2}}{Y_{t+1}}-1\right)\right]=0.00014256.$$

Hence,

$$corr\left(\frac{Y_{t+1}}{Y_t}-1,\frac{Y_{t+2}}{Y_{t+1}}-1\right)=\frac{0.00014256-0.018^2}{0.036^2}=-0.14.$$

- Now we compute P_t^f:

$$P_t^f=E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right]$$

If state 1 at $t\Longrightarrow \underbrace{P^{f1}}_{\text{Price when today is "good"}}=\beta(0.43)(1.054)^{-\sigma}+\beta(0.57)(0.982)^{-\sigma}=0.929248$

If state 2 at $t\Longrightarrow P^{f2}=\beta(0.57)(1.054)^{-\sigma}+\beta(0.43)(0.982)^{-\sigma}=0.911048$

- **Result**:
  - If this year (t) is "good" (t is state 1), the risk-free rate between this year (t) and the next ($t+1$) is:

$$1+r^{f1}=\frac{1}{P^{f1}}=\frac{1}{0.929248}=1.076139$$

  - If this year is "bad" (t is state 2):

$$1+r^{f2}=\frac{1}{P^{f2}}=\frac{1}{0.911048}=1.097637$$

- Try other numbers:

	$\sigma=1$	$\sigma=1$	$\sigma=2$	$\sigma=2$
	$\beta=0.95$	$\beta=0.99$	$\beta=0.95$	$\beta=0.99$
r^{f1}	6.5%	2.2%	**7.6%**	3.3%
r^{r2}	7.6%	3.2%	**9.8%**	5.3%
average	7.1%	2.7%	8.7%	4.3%

- **Implications of the fundamental equation on risk-free rate:**

  1. If this year is "good", then low risk-free rate. "Bad" -> high risk-free rate.

     Intuition: If this year is "good" –> Next year is more likely to be "bad" due to negative autocorrelation of GDP growth. Agents want to save more due to **precautionary motives**. More demand for risk-free assets. Hence, P^f is higher. r^f is lower.

     If "bad" times are expected, low risk-free rate.

  2. Average risk-free rate is around 8-9%.

     This does not match with the annual interest rate implied by the U.S. treasury bills, about 0.5% since 1945.

     Risk-free Rate Puzzle: Risk-free rate implied by the theroy is too high.

2. Stock Return

- **Data**: "Stocks in Aggregate" in EconS 320
- If you "own" this economy, the payoff is the endowment, Y_t, at each period t.
- What is the price of this economy or "wealth portfolio" (market portfolio)?
- That is, what is the price of all companies in this economy? Roughly, this is the value of the stock market. (Question: Human capital also contributes to the output of the economy. How should we deal with human capital? Not enough research has been done on this important research question.)
- **Set-Up**:

  ————+—————————————————-+-

  t t+1

  At the beginning of each period: Dividends paid

  At the end of each period: Assets traded

- From (1),

$$\underbrace{P_t^W}_{\text{Today you pay this}} = E_t[m_{t+1}(\underbrace{Y_{t+1}}_{\text{Tomorrow you get GDP}} + \underbrace{P_{t+1}^W}_{\text{and sell the wealth portfolio}})]$$

  Hence, under CRRA,

$$P_t^W = E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}(Y_{t+1} + P_{t+1}^W)\right]$$

- We can't deal with it — $\{Y_t\}$ grows forever, so P_t^W will also grow forever. We want to make it "stable".

- Hence, divide both sides by Y_t:

$$\begin{aligned}\frac{P_t^W}{Y_t} &= E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\left(\frac{Y_{t+1}}{Y_t}+\frac{P_{t+1}^W}{Y_{t+1}}\frac{Y_{t+1}}{Y_t}\right)\right] \\ &= E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma}\left(1+\frac{P_{t+1}^W}{Y_{t+1}}\right)\right]\end{aligned}$$

- Define $P_t^* = P_t^W/Y_t$. (This is the **price-dividend ratio**, or **P/D ratio**.) Then,

$$P_t^* = E_t\left[\beta\left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma}(1+P_{t+1}^*)\right]$$

- Guess: $\{P_t^*\}$ is P^{*1} if the **current** state is #1 and P^{*2} if the **current** state is 2. That is,

$$\begin{aligned}P^{*1} &= \beta(0.43)(1.054)^{1-\sigma}(1+P^{*1}) + \beta(0.57)(0.982)^{1-\sigma}(1+P^{*2}) \\ P^{*2} &= \beta(0.57)(1.054)^{1-\sigma}(1+P^{*1}) + \beta(0.43)(0.982)^{1-\sigma}(1+P^{*2})\end{aligned}$$

- Hence,

$$\begin{aligned}\begin{bmatrix} P^{*1} \\ P^{*2}\end{bmatrix} &= \underbrace{\begin{bmatrix} \beta(0.43)(1.054)^{1-\sigma} & \beta(0.57)(0.982)^{1-\sigma} \\ \beta(0.57)(1.054)^{1-\sigma} & \beta(0.43)(0.982)^{1-\sigma}\end{bmatrix}}_{\equiv A}\begin{bmatrix} 1+P^{*1} \\ 1+P^{*2}\end{bmatrix} \\ &= A\begin{bmatrix} 1 \\ 1\end{bmatrix} + A\begin{bmatrix} P^{*1} \\ P^{*2}\end{bmatrix}\end{aligned}$$

- Hence,

$$(I-A)\begin{bmatrix} P^{*1} \\ P^{*2}\end{bmatrix} = A\begin{bmatrix} 1 \\ 1\end{bmatrix}.$$

- Hence,

$$\begin{bmatrix} P^{*1} \\ P^{*2}\end{bmatrix} = (I-A)^{-1}A\begin{bmatrix} 1 \\ 1\end{bmatrix}$$

  if $(I-A)$ is invertible.

- Using $\beta = 0.95$ and $\sigma = 2$,

$$\begin{bmatrix} P^{*1} \\ P^{*2}\end{bmatrix} = \begin{bmatrix} 14.2680 \\ 14.1436\end{bmatrix}$$

- Therefore, the price of the wealth portfolio is slightly higher than 14 times the current endowment (GDP).
- We have obtained the price. What about the **returns**?:

$$1 + r_{t+1}^W = \frac{Y_{t+1} + P_{t+1}^W}{P_t^W}$$

- There are four scenarios:
- Scenario 1: "Good" in t, "Good" in $t+1$: The realized return is

$$\begin{aligned}
1 + r_{t+1}^W &= \frac{Y_{t+1} + 14.2680 Y_{t+1}}{14.2680 Y_t} \\
&= \frac{1 + 14.2680}{14.2680} \frac{Y_{t+1}}{Y_t} \\
&= \frac{1 + 14.2680}{14.2680} 1.054 \\
&= 1.1279
\end{aligned}$$

- Scenario 2: "Good" in t, "Bad" in $t+1$:

$$\begin{aligned}
1 + r_{t+1}^W &= \frac{Y_{t+1} + 14.1436 Y_{t+1}}{14.2680 Y_t} \\
&= \frac{1 + 14.1436}{14.2680} \frac{Y_{t+1}}{Y_t} \\
&= \frac{1 + 14.1436}{14.2680} 0.982 \\
&= 1.0423
\end{aligned}$$

- Scenario 3: "Bad" in t, "Good" in $t+1$:

$$\begin{aligned}
1 + r_{t+1}^W &= \frac{Y_{t+1} + 14.2680 Y_{t+1}}{14.1436 Y_t} \\
&= \frac{1 + 14.2680}{14.1436} \frac{Y_{t+1}}{Y_t} \\
&= \frac{1 + 14.2680}{14.1436} 1.054 \\
&= 1.1378
\end{aligned}$$

- Scenario 4: "Bad" in t, "Bad" in $t+1$:

$$\begin{aligned} 1 + r_{t+1}^W &= \frac{Y_{t+1} + 14.1436 Y_{t+1}}{14.1436 Y_t} \\ &= \frac{1 + 14.1436}{14.1436} \frac{Y_{t+1}}{Y_t} \\ &= \frac{1 + 14.1436}{14.1436} 0.982 \\ &= 1.0514 \end{aligned}$$

- **Implications**:

- **A. Average Equity Premium**: First, what is $E[r_{t+1}^W]$, unconditional expected return?

- Trick:

```
                    +——> State 1 [Prob: 0.5*0.43] r^W_{t+1} = 0.1279
[50%] State 1 —+
                    +——> State 2 [Prob: 0.5*0.57] r^W_{t+1} = 0.0423
                    +——> State 1 [Prob: 0.5*0.57] r^W_{t+1} = 0.1378
[50%] State 2 —+
                    +——> State 2 [Prob: 0.5*0.43] r^W_{t+1} = 0.0514
```

Hence,

$$E\left[r_{t+1}^W\right] = 0.089878.$$

- The unconditional average of return on the wealth portfolio is 9.0%.

- But risk-free rate is predicted to be at 8.7%. $\longrightarrow$ A predicted level of the **equity premium** is only 0.3%.

- This is the premium paid for the risk of the wealth portfolio.

- Reality: The stock market return is about 8% on average. The equity premium is about 7.5%.

- **Conclusion**: The prediction on the equity premium is too low. This is known as the **equity premium puzzle**.

- Two puzzles, risk-free rate puzzle and equity premium puzzle, are probably related.

- **B. Cyclicality of Stock Returns:**

- The realized stock returns are higher at "good" state.

  "Good" pays 12.8% (scenario 1) and 13.8% (scenario 3).

  "Bad" pays only 4.2% (scenario 2) and 5.1% (scenario 4).

  Intuition: Simply, "good" state pays more dividend.

- Returns from previous "bad" state are higher.

  From "Bad": 13.8% (scenario 3) and 5.1% (scenario 4)

  From "Good": 12.8% (scenario 1) and 4.2% (scenario 2)

  Why? Because at "bad", the price (normalized to dividend) is cheaper (i.e., lower P/D ratio). Also, from "bad", the stock is more likely to pay higher dividends in the next period due to negative autocorrelation of endowment growth.

- **C. Return Volatility**

- We now obtain $sd(r^W_{t+1})$, unconditional standard deviation of r^W_{t+1}.

- This is a square root of

$$var(r^W_{t+1}) = E[(r^W_{t+1})^2] - \underbrace{E[r^W_{t+1}]^2}_{0.089878^2}$$

- What is $E[(r^W_{t+1})^2]$?

- Trick:

+——> State 1 [Prob: 0.5*0.43] $(r^W_{t+1})^2 = 0.1279^2$

[50%] State 1 —+

+——> State 2 [Prob: 0.5*0.57] $(r^W_{t+1})^2 = 0.0423^2$

+——> State 1 [Prob: 0.5*0.57] $(r^W_{t+1})^2 = 0.1378^2$

[50%] State 2 —+

+——> State 2 [Prob: 0.5*0.43] $(r^W_{t+1})^2 = 0.0514^2$

Hence,

$$E\left[(r^W_{t+1})^2\right] = 0.010007.$$

- Hence,

$$sd(r^W_{t+1}) = \sqrt{0.010007 - 0.089878^2} = 0.043918.$$

- The standard deviation predicted by the model is only 4.4%. Data: 18%.

- This is known as the **excess volatility**.

- **D. P/D Ratio (inverse of Dividend Yield)**
- Recall:

$$\begin{bmatrix} P^{*1} \\ P^{*2} \end{bmatrix} = \begin{bmatrix} 14.2680 \\ 14.1436 \end{bmatrix}$$

  - (i) **Average**: Average is about 14.2. This is lower than 30.0, reported in Menzly, Santos and Veronesi (2004).
  - (ii) **Cyclicality**: $P^{*1} > P^{*2}$. The stock price relative to divident is higher at good times. Why? Good time (state 1) $\longrightarrow$ Next period is more likely to be bad $\longrightarrow$ (a) Save now for precautionary motives. Demand $\uparrow$. (b) This asset's expected dividend for the next period is lower. Demand $\downarrow$. Here, (i) dominates.
  - (iii) **Predictive Power**:

Period-t state	Period-t P/D ratio	Expected $(t+1)$-Return
"Good"	14.2680	$0.43 \times 12.8\% + 0.57 \times 4.2\% = 7.9\%$
"Bad"	14.1436	$0.57 \times 13.8\% + 0.43 \times 5.1\% = 10.1\%$

  - The P/D ratio has a predictive power. If P/D ratio is high, then expected return is low, and vice versa.
  - cf. Menzly, Santos and Veronesi (2004, p.31, Table 5): Pooled regression of industry-level returns on lagged dividend yields. The slope is 2.9.

3. What May Be Wrong?

- Issues: Risk-free rate puzzle, equity-premium puzzle, slope of returns on lagged dividend yield, excess volatility, ...
- Potential solutions: limited participation, non-stock assets (such as human capital), rare disasters, the recursive utility, habit, ...

EXERCISES

1. Descriptive Analyses:

   (a) Problem 36 in my Money and Banking Exercise Questions, which is available at http://user.chol.com/~estudiar/English/320_10_bank.pdf.

   (b) Problem 67 in my Money and Banking Exercise Questions, which is available at http://user.chol.com/~estudiar/English/320_10_bank.pdf.

   (c) Problem 68 in my Money and Banking Exercise Questions, which is available at http://user.chol.com/~estudiar/English/320_10_bank.pdf.

2. Consider the following four (excess) returns:

   (a) The real treasury bill rate (a proxy for the risk-free rate)

   (b) The excess return of the stock market to treasury bill: The value-weight return on all NYSE, AMEX, and NASDAQ stocks minus treasury bill rate

   (c) The excess return of smaller stocks (of bottom 50% in size) to larger stocks (of top 50% in size), a.k.a. SMB (Small Minus Big)

   (d) The excess return of value stocks (with top 30% book-to-market ratios) to growth stocks (with bottom 30% book-to-market ratios), a.k.a. HML (High Minus Low).

   Visit http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html, which is Kenneth French's data library. See "Details" next to "Fama/French Factors" for further information on the above four (excess) returns. Monthly (excess) returns (%) for all four above are reported in "Fama/French Factors". For (a), subtract "RF" from the CPI inflation rate using Bureau of Labor Statistics data. For (b), (c) and (d), use "Mkt-RF", "SMB" and "HML" directly. (You do not have to worry about inflation here. Why?)

   Use the data from January 1946 to the last month available. Regarding the above four (excess) returns, report the sample annual means during the following periods, based on NBER business cycle data:

   (a) From: One year before the peak, To: Peak

   (b) From: Peak, To: One year after the peak

   (c) From: One year before trough, To: Trough

(d) From: Trough, To: One year after Trough

Note: This is like re-doing Problem 54 in my Money and Banking Exercise Questions with four other portfolios.

3. Provide your prediction on the average annual real U.S. stock return over the three years, from January 2010 to December 2012, for a general reader (not holding Ph.D. in economics or finance).

4. True or False? Explain: Since investors use all available information in their investment decisions, no financial variables currently observable predict (future) returns on stocks, bonds or other assets.

   Answer: *FALSE - Empirically, (i) A downward-sloping yield curve tends to predict a recession after 1 year, in which the stock returns tend to be low. (ii) A low P/D ratio (or a high dividend yield) tends to predict high stock returns in the medium run.*

5. Answer the following questions.

   (a) Provide at least one example of a financial variable that has a predictive power on the future stock market returns. According to the data, when the observed value of this financial variable increases, what tends to happen to the future stock market returns?

   Answer: *Dividend-Price ratio. When it is high, the next 7-year returns tend to be high.*

   (b) Provide at least one example of a financial variable that has a predictive power on the future business cycle. According to the data, when _______________, it is likely that the economy will be in a recession after one year.

   Answer: *"the slope of yield curve (especially 3M vs. 10Y) is downward sloping"*

6. Provide your research proposal on the following question: A large part of the U.S. government bonds are owned by foreign governments. If foreign governments sell all their holdings all of a sudden, what will be a new interest rate on U.S. government bonds? Your research is expected to provide a *number*.

   Guideline: I look for the following three items, almost equally weighted in grading. (i) Set up a theoretical model, which is preferably an updated version of the one discussed in class. (ii) Discuss which data you will need and how you will calibrate or estimate the model. (iii) Discuss what your research is likely to be. Discuss any preliminary results. What do you expect can and cannot be answered? What are the shortcomings of your approach?

7. Consider an endowment economy with a representative consumer, a single perishable physical good and a complete financial market (as in the class). The consumer maximizes at time 0

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right] = E_0\left[\sum_{t=0}^{\infty}\beta^t \frac{C_t^{1-\sigma}}{1-\sigma}\right],$$

where $0 < \beta < 1$, $\sigma > 0$ and $\sigma \neq 1$. (And $u(c) = \log c$ if $\sigma \longrightarrow 0$.) There are two "states." The growth of endowment is

$$\frac{Y_{t+1}}{Y_t} - 1 = \begin{cases} 0.054 \text{ (state 1: "good")} \\ -0.018 \text{ (state 2: "bad")} \end{cases}$$

Assume a Markov chain with a transition matrix:

$$P = \begin{pmatrix} \text{prob}_{1\to 1} & \text{prob}_{1\to 2} \\ \text{prob}_{2\to 1} & \text{prob}_{2\to 2} \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \begin{pmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{pmatrix}$$

as in Mehra and Prescott (1985). Obtain the model's prediction on the risk-free rate and the rate of return on the wealth portfolio. Complete the following table. Discuss the risk-free rate puzzle and the equity premium puzzle.

	Data	$\sigma = 2$	2	100	100
		$\beta = 0.95$	0.99	0.95	0.99
Average risk-free rate (A)	<1%*	8.7%			
Average return on wealth portoflio (B)	about 8%**	9.0%			
Average equity premium (B−A)	about 7%	0.3%			

*: Annual average rate of return on the U.S. 3-month treasury since 1945

**: Annual average rate of return on the U.S. stock market since 1945

Answer:

	Data	$\sigma = 2$	2	100	100
		$\beta = 0.95$	0.99	0.95	0.99
Average risk-free rate (A)	<1%*	8.7%	4.3%	-65.8%	-67.2%
Average return on wealth portoflio (B)	about 8%**	9.0%	4.6%	-48.6%	-50.0%
Average equity premium (B−A)	about 7%	0.3%	0.3%	17.3%	17.2%

8. What is the volatility (or unconditional standard deviation) of the risk-free rate, predicted by the model considered in class? Is it consistent with the data?

Answer: *We have* $r^{f1} = 0.076139$ *and* $r^{f2} = 0.097637$ *for two current states, 1* *and 2. The unconditional probability of each state is 50%. Hence,*

$$\begin{aligned} sd(r_t^f) &= \sqrt{var(r_t^f)} = \sqrt{E[(r_t^f)^2] - E[r_t^f]^2} \\ &= \sqrt{\frac{1}{2}[0.076139^2 + 0.097637^2] - 0.087^2} \\ &= 0.009801303 \end{aligned}$$

So it is about 1%. Lower than 2-3%, empirically observed for the U.S. govt bonds.

9. *(Rare Disasters)* In this problem, you will consider how a possible occurrence of "rare disasters" in aggregate endowment can affect asset prices, as explored by Rietz (1988) and Barro (2006). Consider a closed, endowment economy with a representative consumer, a single perishable physical good and a complete financial market. The consumer maximizes at period 0

$$E_0\left[\sum_{t=0}^{\infty}\beta^t \frac{C_t^{1-\sigma}}{1-\sigma}\right],$$

where $0 < \beta < 1$, $\sigma > 0$ and $\sigma \neq 1$, and C_t is the consumption at period t. The endowment growth is

$$\frac{Y_{t+1}}{Y_t} = \begin{cases} \exp(g) \text{ with probability } 1-p \text{ ("no disaster")} \\ \exp(g)B \text{ with probability } p \text{ ("disaster")} \end{cases}$$

Here, Y_t is the endowment at period t. Also, g, B and p are constant. Notice that B reflects an endowment decrease in case of a disaster. For example, if $B = 0.7$, then the endowment falls by 30%. A disaster occurs with probability p at each period. The occurrence of the disaster is independent and identically distributed over time.

(a) What is your economic interpretation of g? Write in plain English.

Answer: *It is the normal-times growth rate, such as 2% per year for the U.S. Taking logs in* $\frac{Y_{t+1}}{Y_t} = \exp(g)$, *we have* $\log Y_{t+1} - \log Y_t = g$. *The LHS is the growth rate of* Y_t.

(b) The fundamental equation of asset pricing is $P_0^j = E_0[m_1 X_1^j]$, where P_0^j is the price of one-period asset j at period 0, m_1 is the stochastic discount factor between periods 0 and 1 for any one-period asset, and X_1^j is the payoff of asset j at period 1. Obtain m_1 for the given preferences.

Answer: $m_1 = \beta u'(C_1)/u'(C_0) = \beta(C_1/C_0)^{-\sigma}$.

(c) Obtain a formula for a one-period risk-free rate between periods 0 and 1, denoted by r_0^f. Obtain the value of r_0^f by calibration, using $B = 0.7$, $p = 0.008$, $\exp(g) = 1.0202$, $\sigma = 10$, and $\beta = 0.95$.

Answer: *We have*

$$\frac{1}{1+r_t^f} = E_0\left[\beta(C_1/C_0)^{-\sigma}\right] = (1-p)\beta[\exp(g)]^{-\sigma} + p\beta[\exp(g)B]^{-\sigma}$$
$$= \beta[\exp(g)]^{-\sigma}[1-p+pB^{-\sigma}] = 0.991851$$

So

$$r_0^f = 0.008215 = 0.8\%.$$

(d) The wealth portfolio provides the endowment as dividend at each period. That is, if one pays P_0^W units of physical goods to purchase a unit of this asset, it delivers Y_1 units of physical goods at period 1, Y_2 units at period 2, etc. Calibrate the price-dividend ratio of this asset at period 0, i.e., P_0^W/Y_0, using the numbers given in (c).

Answer: *We have*

$$P_0^W = E_0\left[\beta(Y_1/Y_0)^{-\sigma}(P_1^W + Y_1)\right].$$
$$\text{So } \frac{P_0^W}{Y_0} = E_0\left[\beta\left(\frac{Y_1}{Y_0}\right)^{1-\sigma}\left(\frac{P_1^W}{Y_1}+1\right)\right].$$
$$\text{So } P_0^{W*} = E_0\left[\beta\left(\frac{Y_1}{Y_0}\right)^{1-\sigma}\left(P_1^{W*}+1\right)\right].$$

Here, $P_0^{W*} \equiv \frac{P_0^W}{Y_0}$*. The aggregate states are realized i.i.d. over time. So there is no reason why* P_0^{W*} *should be different from* P_1^{W*}*. Note: If you do not believe it, write it as a two-state problem, where* NO *is "no disaster" and* $DISA$ *is "disaster" :*

$$P_{NO}^{W*} = (1-p)\beta\left(\exp(g)\right)^{1-\sigma}\left(P_{NO}^{W*}+1\right) + p\beta\left(\exp(g)B\right)^{1-\sigma}\left(P_{DISA}^{W*}+1\right),$$
$$P_{DISA}^{W*} = (1-p)\beta\left(\exp(g)\right)^{1-\sigma}\left(P_{NO}^{W*}+1\right) + p\beta\left(\exp(g)B\right)^{1-\sigma}\left(P_{DISA}^{W*}+1\right).$$

So $P_{NO}^{W*} = P_{DISA}^{W*}$*. Hence,*

$$P^{W*} = (1-p)\beta\left(\exp(g)\right)^{1-\sigma}\left(P^{W*}+1\right) + p\beta\left(\exp(g)B\right)^{1-\sigma}\left(P^{W*}+1\right)$$
$$= \left(P^{W*}+1\right)\beta\left(\exp(g)\right)^{1-\sigma}\left[1-p+pB^{1-\sigma}\right]$$

So

$$P^{W*} = \frac{\beta\left(\exp(g)\right)^{1-\sigma}\left[1-p+pB^{1-\sigma}\right]}{1-\beta\left(\exp(g)\right)^{1-\sigma}\left[1-p+pB^{1-\sigma}\right]} = 17.00807.$$

(e) Calibrate the expected return on the wealth portfolio, $E_0[r_1^W]$, where $1 + r_1^W \equiv (Y_1 + P_1^W)/P_0^W$, using the numbers given in (c). Discuss whether an introduction of rare disasters can improve the model's predictions on the risk-free rate and equity premium.

Answer: *We have*

$$
\begin{aligned}
1 + r_1^W &\equiv \frac{Y_1 + P_1^W}{P_0^W} = \frac{Y_1/Y_0 + (P_1^W/Y_1)(Y_1/Y_0)}{P_0^W/Y_0} \\
&= \frac{Y_1}{Y_0}\frac{1 + P^{W*}}{P^{W*}}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E[r_1^W] &= \frac{1 + P^{W*}}{P^{W*}}\left[(1-p)\exp(g) + p\exp(g)B\right] - 1 \\
&= \frac{1 + P^{W*}}{P^{W*}}\exp(g)\left[1 - p + pB\right] - 1 \\
&= 0.077592.
\end{aligned}
$$

Rare disasters potentially improve the performance of the model. Risk-free rate is 0.8%, close to the data. The equity premium is $7.8\% - 0.8\% = 6.9\%$*, close to the data.*

Continuous-Time Stochastic Models

References

- Cochrane, J. H. (2005), Asset Pricing, Revised Edition, Princeton University Press, Princeton, NJ, Chapter 1 and Appendix.

We discussed asset-pricing theories under discrete-time set-up. Some economists use another language: continuous time. An advantage for continuous time is that it often provides algebraically useful techniques that are not available for discrete time. A disdavantage is that it takes time to be familiar with it (just like with any language). Whenver I see the continuous time, I think of it after transforming into the discrete time in my mind.

This is an **informal introductory summary** of continuous-time stochastic processes for 1st-year economics and finance students. Formal discussions can be found at other courses in economics, mathematics and statistics.

1. Brownian Motion

- Consider a **random walk** in discrete time:
- $z_{t+1} - z_t = \varepsilon_{t+1}$,

  $z_{t+2} - z_{t+1} = \varepsilon_{t+2}$,

  ...

  where ε_τ˜iid $N(0,1)$ for all $\tau = t+1, t+2, ...$
- This implies

  $var(z_{t+1} - z_t) = 1$,

  $var(z_{t+2} - z_t) = var[\varepsilon_{t+1} + \varepsilon_{t+2}] = var(\varepsilon_{t+1}) + var(\varepsilon_{t+2}) = 2$,

  ...,

  $var(z_{t+k} - z_t) = k$ for some integer $k > 0$.
- So $z_{t+k} - z_t = \varepsilon_{t+1} + \varepsilon_{t+2} + ... + \varepsilon_{t+k}$˜$N(0, k)$.

- The idea is to extend this model (random walk) into the continuous time (**Brownian Motion**). That is, write
$$z_{t+\Delta} - z_t \tilde{} N(0, \Delta)$$
where Δ is "some small number."

- We write Δ more formally as "dt" (an infinitesimal increase in time).

- And we write
$$dz_t \equiv \lim_{\Delta \to 0} z_{t+\Delta} - z_t \tilde{} N(0, dt)$$

- So $\boxed{dz_t \text{ in continuous time is like } z_{t+1} - z_t = \varepsilon_{t+1} \text{ in discrete time}}$.

- **Property 1**: $\boxed{E_t\left[dz_t\right] = 0}$ just like $\boxed{E_t[z_{t+1} - z_t] = E_t[\varepsilon_{t+1}] = 0}$.

- **Property 2**: $\boxed{var_t\left[dz_t\right] = dt}$ just like $\boxed{var_t[z_{t+1} - z_t] = var_t[\varepsilon_{t+1}] = 1}$ or
$$\boxed{var_t[z_t + k - z_t] = var_t[\varepsilon_t + 1 + \varepsilon_t + 2 + ... + \varepsilon_t + k] = k}.$$

- **Property 3A**: $\boxed{dz_t \tilde{} N(0, dt)}$ just like $\boxed{\varepsilon_t \tilde{} N(0, 1)}$.

- Just like $z_{t+k} - z_t = \sum_{j=1}^{k} \varepsilon_{t+j}$, we write
$$z_{t+s} - z_t = \int_0^s dz_t,$$
where $\int$ here is called a "stochastic integral." (In words, the gap between z_{t+s} and z_t consists of a lot of pieces of dz_t.)

- **Property 3B**: $\boxed{\int_0^s dz_t \tilde{} N(0, s)}$ just like $\boxed{z_{t+k} - z_t = \sum_{j=1}^{k} \varepsilon_{t+j} \tilde{} N(0, k)}$.

- Summary so far:

Discrete Time	*Continuous Time*
Random Walk	Brownian Motion
$z_{t+1} - z_t = \varepsilon_{t+1}$ (stochastic drift)	dz_t
1 (smallest time interval)	dt (or "small Δ")
$\varepsilon_{t+1} \tilde{} N(0, 1)$	$dz_t \tilde{} N(0, dt)$
$z_{t+k} - z_t = \sum_{j=1}^{k} \varepsilon_{t+j} \tilde{} N(0, k)$	$\int_0^s dz_t \tilde{} N(0, s)$

- Note: "d" implies an "infinitesimal increase." In terms of problem solving, I find it useful to treat "dt" and "dz_t" as one letter. In other words, whenever you see "d" in continuous-time model in economics, consider it as completely attached to t, z_t, etc.

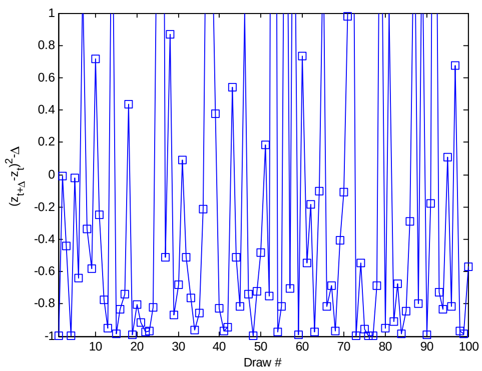
- dz_t is one letter. So dz_t^2 implies $(dz_t)^2$, not $d(z_t)^2$.
- Property 2 says

$$var_t[dz_t] = E_t[(dz_t)^2] - \underbrace{E_t[dz_t]}_{=0}^2 = E_t[(dz_t)^2] \equiv E_t[dz_t^2] = dt$$

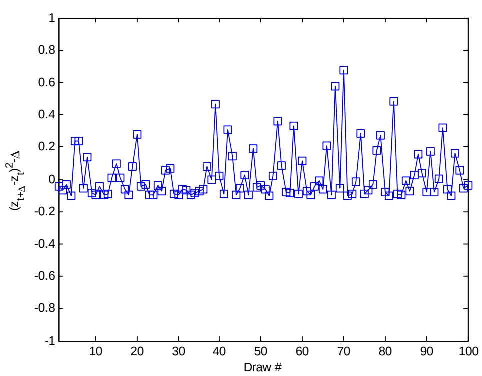
- So $E_t[dz_t^2] = dt$. But we now have a shocking theorem:
- **Theorem 1**: $dz_t^2 = dt$.
- (Proof? Take Stochastic Calculus.)
- So dz_t is stochastic, but if you square it, the randomness disappears. How can we understand it?
- Theorem 1 implies $dz_t^2 - dt = 0$.
- Recall $dz_t \equiv \lim_{\Delta\to 0} z_{t+\Delta} - z_t$.
- Let's see how $(z_{t+\Delta} - z_t)^2 - \Delta$ reacts as we approach $\Delta \to 0$.
- Strategy: For $\Delta = 1$, draw 100 values for $z_{t+\Delta} - z_t \tilde{} N(0, \Delta)$ and compute $(z_{t+\Delta} - z_t)^2 - \Delta$ for each of them.
- For $\Delta = 1/10$, repeat.
- For $\Delta = 1/10^2$, repeat.
- ...
- I have this MatLab code:
  - clear all;
  - close all;
  - Delta=1/10^0; % You change this number: Try 1, 1/10, 1/100, etc.
  - Number=100; % Number of draws.
  - Result=[]; % Vector of results, initially empty.
  - for i=1:Number,
  - Result=[Result,(random('normal',0,sqrt(Delta)))^2-Delta]; % This generates a random number based on N(0,Delta), square it, and then subtract Delta.

- end
- figure(1);
- plot([1:1:Number],Result,'s-');
- axis([1,Number,0,10]);

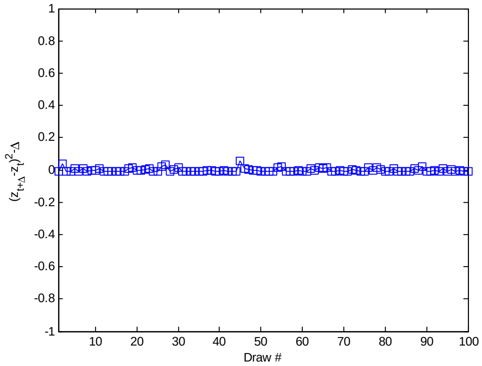
- Results: As $\Delta \to 0$, $(z_{t+\Delta} - z_t)^2 - \Delta$ also converges to 0.
  - (i) When $\Delta = 1$



  -
  - (ii) When $\Delta = 1/10$



  -
  - (iii) When $\Delta = 1/10^2$



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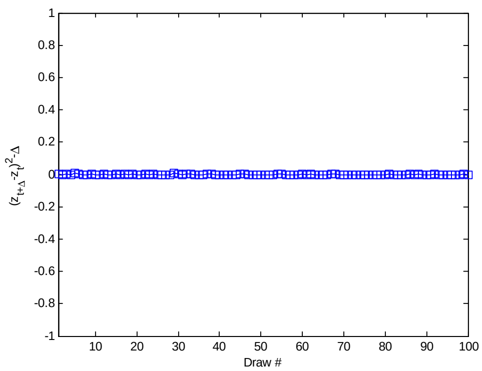
– (iv) When $\Delta = 1/10^3$



–

- **Theorem 2**: $dz_t dt = 0$. Also, $dt^2 = 0$. Also, anything higher$= 0$.
- Note: Again, dt is one unit, so $dt^2 \equiv (dt)^2$.
- A slight update of the above MatLab code will show this.

2. Ito's Lemma

- Consider a discrete-time model:

$$x_{t+1} = \mu + x_t + \sigma\varepsilon_{t+1}$$
$$\text{So } x_{t+1} - x_t = \mu + \sigma\varepsilon_{t+1}$$

- A **diffusion process** takes the form

$$dx_t = \underbrace{\mu dt}_{\substack{\textbf{drift term} \\ \text{(constant drift)}}} + \underbrace{\sigma dz_t}_{\substack{\textbf{diffusion term} \\ \text{(stochastic drift)}}}$$

- Ito's Lemma: Application of Taylor approximation to continuous-time model.

- Given $dx_t = \mu dt + \sigma dz_t$, what is the diffusion process for $y_t = f(x_t, t)$? That is, what is dy_t?

- There are two elements: x_t and t. Note: Here, treat x_t as one letter. Ignore a subscript t, that is, x_t is NOT a function of t.

$$dy_t = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\underbrace{dt^2}_{=0 \text{ by Theorem 2}} + \frac{\partial f}{\partial x_t}dx_t + \frac{1}{2}\frac{\partial^2 f}{\partial x_t^2}dx_t^2 + \frac{\partial^2 f}{\partial t \partial x_t}dx_t dt ...(*)$$

- Digression: This reminds us of the Taylor approximation:

$$g(x) \approx g(\alpha) + (x-\alpha)g'(\alpha) + \frac{1}{2}(x-\alpha)^2 g''(\alpha).$$

For two-dimensional:

$$\begin{aligned} h(x,y) \approx\ & h(\alpha,\beta) + (x-\alpha)\frac{\partial h(\alpha,\beta)}{\partial \alpha} + \frac{1}{2}(x-\alpha)^2\frac{\partial^2 h(\alpha,\beta)}{\partial \alpha^2} \\ & + (y-\beta)\frac{\partial h(\alpha,\beta)}{\partial \beta} + \frac{1}{2}(y-\beta)^2\frac{\partial^2 h(\alpha,\beta)}{\partial \beta^2} + (x-\alpha)(y-\beta)\frac{\partial^2 h(\alpha,\beta)}{\partial \alpha \partial \beta} \end{aligned}$$

So

$$\begin{aligned} h(x,y) - h(\alpha,\beta) \approx\ & (x-\alpha)\frac{\partial h(\alpha,\beta)}{\partial \alpha} + \frac{1}{2}(x-\alpha)^2\frac{\partial^2 h(\alpha,\beta)}{\partial \alpha^2} \\ & + (y-\beta)\frac{\partial h(\alpha,\beta)}{\partial \beta} + \frac{1}{2}(y-\beta)^2\frac{\partial^2 h(\alpha,\beta)}{\partial \beta^2} + (x-\alpha)(y-\beta)\frac{\partial^2 h(\alpha,\beta)}{\partial \alpha \partial \beta} \end{aligned}$$

- Go further from (*):

$$\begin{aligned} dx_t &= \mu dt + \sigma dz_t, \\ dx_t^2 &= (\mu dt + \sigma dz_t)^2 = \mu^2 \underbrace{dt^2}_{=0 \text{ by Theorem 2}} + 2\mu\sigma^2 \underbrace{dt dz_t}_{=0 \text{ by Theorem 2}} + \sigma^2 \underbrace{dz_t^2}_{=dt \text{ by Theorem 1}} = \sigma^2 dt \\ dx_t dt &= (\mu dt + \sigma dz_t)dt = \mu \underbrace{dt^2}_{=0 \text{ by Theorem 2}} + \sigma \underbrace{dt dz_t}_{=0 \text{ by Theorem 2}} = 0 \end{aligned}$$

by Theorem 1 and Theorem 2.

- Therefore,

$$\begin{aligned} dy_t &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_t}[\mu dt + \sigma dz_t] + \frac{1}{2}\frac{\partial^2 f}{\partial x_t^2}\sigma^2 dt \\ &= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_t}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x_t^2}\sigma^2\right]dt + \frac{\partial f}{\partial x_t}\sigma dz_t \text{ (Ito's Lemma)} \end{aligned}$$

- $\boxed{\text{You should be able to \textit{derive} this.}}$

- I don't recommend you to memorize this final formula. This is only one version (although representative) of Ito's Lemma. Any similar application (with one-variable function, etc.) is also considered Ito's Lemma. Here are some examples of such applications:

- **Example 1**: Suppose $\frac{dx_t}{x_t} = \mu dt + \sigma dz_t$. What is $d \log x_t$?

- **Answer**: Define $y_t = f(x_t) = \log x_t$. One element is x_t. Write $(*)$ for one variable x_t only:

$$dy_t = \frac{\partial f}{\partial x_t} dx_t + \frac{1}{2} \frac{\partial^2 f}{\partial x_t^2} dx_t^2$$

- But

$$\begin{aligned}
\left(\frac{dx_t}{x_t}\right)^2 &= (\mu dt + \sigma dz_t)^2 \\
&= \mu^2 \underbrace{dt^2}_{=0 \text{ by Theorem 2}} + \sigma^2 \underbrace{dz_t^2}_{=dt \text{ by Theorem 1}} + 2\mu\sigma \underbrace{dt dz_t}_{=0 \text{ by Theorem 2}} \\
&= 0 + \sigma^2 dt + 0 \\
&= \sigma^2 dt
\end{aligned}$$

by Theorem 1 and Theorem 2.

- Hence,

$$\begin{aligned}
dy_t &= \frac{1}{x_t} dx_t + \frac{1}{2}\left(-\frac{1}{x_t^2}\right) dx_t^2 \\
&= \frac{dx_t}{x_t} - \frac{1}{2}\left(\frac{dx_t}{x_t}\right)^2 \\
&= \mu dt + \sigma dz_t - \frac{1}{2}\sigma^2 dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz_t
\end{aligned}$$

- **Example 2**: Show that $d(x_t y_t) = x_t dy_t + y_t dx_t + dx_t dy_t$.

- **Answer**: Two elements are x_t and y_t.

$$\begin{aligned}
d(x_t y_t) &= \frac{\partial (x_t y_t)}{\partial x_t} dx_t + \frac{1}{2} \frac{\partial^2 (x_t y_t)}{\partial x_t^2} dx_t^2 + \frac{\partial (x_t y_t)}{\partial y_t} dy_t + \frac{1}{2} \frac{\partial^2 (x_t y_t)}{\partial y_t^2} dy_t^2 + \frac{\partial^2 (x_t y_t)}{\partial x_t \partial y_t} dx_t dy_t \\
&= y_t dx_t + \frac{1}{2} 0 dx_t^2 + x_t dy_t + \frac{1}{2} 0 dy_t^2 + 1 dx_t dy_t \\
&= y_t dx_t + x_t dy_t + dx_t dy_t
\end{aligned}$$

3. Fundamental Equation in Continuous Time

- Contents:
  - A. Modelling
  - B. Continuous-Time Fundamental Equation (corresponding to $P_t^j = E_t\left[\sum_{\tau=1}^{\infty} m_{t,t+\tau} D_{t+\tau}^j\right]$)
  - C. Continuous-Time Fundamental Equation (corresponding to $P_t^j = E_t[m_{t+1}(D_{t+1}^j + P_{t+1}^j)]$)
  - D. Continuous-Time Fundamental Equation (corresponding to $1 = E_t[m_{t+1}(1 + r_{t+1}^j)]$)

- **A. Modelling**

- In discrete time, you buy an asset (price: P_t^j) at period t. This asset pays dividend *after* that. So the first dividend paid is D_{t+1}^j.

  t $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $t+1$

  +————————————————+

  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ D_{t+1}^j

- In continuous time, you buy an asset (price: P_t^j) at period t.

- Now the time is continous, so it immediately starts to pay dividends from $t + \Delta$ (or $t + dt$), and it continues to pay at every moment.

- The "rate" of this dividend payment is D_t^j, implying that the flow of dividends for *one* period is accumulated to D_t^j.

- So for in infinitesmal time interval dt, the dividend paid is $D_t^j dt$.

  t $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $t+1$

  +————————————————+

  <————————————————>

  D_t^j ("rate" of dividend payment)

  $D_t^j dt$ (dividend paid for infinitesmal time)

-

	Discrete Time	*Continuous Time*
Price	P_t^j	P_t^j
Dividend	D_{t+1}^j for 1 period	$D_t^j dt$ for infinitesmal (dt)
Consumer's preferences	$E_t\left[\sum_{\tau=1}^{\infty} \beta^\tau u(C_{t+\tau})\right]$	$E_t\left[\int_0^\infty e^{-\delta\tau} u(C_{t+\tau}) d\tau\right]$

- **B. Continuous-Time Fundamental Equation (corresponding to** $P_t^j = E_t\left[\sum_{\tau=1}^{\infty} m_{t,t+\tau} D_{t+\tau}^j\right]$**)**

- Recall that we solved the discrete-time consumer's problem for an infinitely-lived asset as follows:

$$\text{(marginal cost of holding the asset)} = \text{(marginal benefit of holding the asset)}$$

$$u'(C_t)P_t^j = E_t\left[\sum_{\tau=1}^{\infty} \beta^\tau u'(C_{t+\tau}) D_{t+\tau}^j\right]$$

$$\text{So } P_t^j = E_t\left[\sum_{\tau=1}^{\infty} \underbrace{\beta^\tau \frac{u'(C_{t+\tau})}{u'(C_t)}}_{\equiv m_{t,t+\tau}} D_{t+\tau}^j\right]$$

$$\text{So } P_t^j = E_t\left[\sum_{\tau=1}^{\infty} m_{t,t+\tau} D_{t+\tau}^j\right]$$

- The continuous-time counterpart is

$$\text{(marginal cost of holding the asset)} = \text{(marginal benefit of holding the asset)}$$

$$u'(C_t)P_t^j = E_t\left[\int_0^{\infty} e^{-\delta\tau} u'(C_{t+\tau}) D_{t+\tau}^j d\tau\right]$$

- Define $\Lambda_{t+\tau} \equiv e^{-\delta(t+\tau)} u'(C_{t+\tau})$. This means $\Lambda_t = e^{-\delta t} u'(C_t)$. Then,

$$\underbrace{e^{-\delta t} u'(C_t)}_{=\Lambda_t} P_t^j = E_t\left[\int_0^{\infty} \underbrace{e^{-\delta(t+\tau)} u'(C_{t+\tau})}_{=\Lambda_{t+\tau}} D_{t+\tau}^j d\tau\right]$$

$$\Lambda_t P_t^j = E_t\left[\int_0^{\infty} \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right]$$

- This is the **fundamental equation** in the continuous time.

- **C. Continuous-Time Fundamental Equation (corresponding to** $P_t^j = E_t[m_{t+1}(D_{t+1}^j + P_{t+1}^j)]$**)**

- Manipulate the fundamental equation as follows. Use "small" Δ:

$$\begin{aligned}
\Lambda_t P_t^j &= E_t\left[\int_0^\infty \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right] \\
&= E_t\left[\int_0^\Delta \Lambda_{t+\tau} D_{t+\tau}^j d\tau + \int_\Delta^\infty \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right] \\
&= \underbrace{E_t\left[\int_0^\Delta \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right]}_{\text{(A)}} + \underbrace{E_t\left[\int_\Delta^\infty \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right]}_{\text{(B)}}
\end{aligned}$$

- (B):

$$\begin{aligned}
E_t\left[\int_\Delta^\infty \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right] &= E_t\left[\underbrace{E_{t+\Delta}\left[\int_\Delta^\infty \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right]}_{=\Lambda_{t+\Delta} P_{t+\Delta}^j \text{ by fundamental eq.}}\right] \text{ (law of iterated expectations)} \\
&= E_t\left[\Lambda_{t+\Delta} P_{t+\Delta}^j\right]
\end{aligned}$$

- (A): For small Δ, we can apply this approximation:

$$E_t\left[\int_0^\Delta \Lambda_{t+\tau} D_{t+\tau}^j d\tau\right] \approx \Lambda_t D_t^j \Delta$$

- Hence,

$$\begin{aligned}
\Lambda_t P_t^j &= \text{(A)} + \text{(B)} \\
&= \Lambda_t D_t^j \Delta + E_t\left[\Lambda_{t+\Delta} P_{t+\Delta}^j\right]
\end{aligned}$$

- Hence,

$$\Lambda_t D_t^j \Delta + E_t\left[\Lambda_{t+\Delta} P_{t+\Delta}^j - \Lambda_t P_t^j\right] = 0$$

- Write with "d" and "dt":

$$\Lambda_t D_t^j dt + E_t\left[d(\Lambda_t P_t^j)\right] = 0$$

- This is another version of the **fundamental equation**. This corresponds to $P_t^j = E_t[m_{t+1}(D_{t+1}^j + P_{t+1}^j)]$ in discrete time.

- We can go further. From Example 2 of Ito's Lemma, we have $d(\Lambda_t P_t^j) = P_t^j d\Lambda_t + \Lambda_t dP_t^j + d\Lambda_t dP_t^j$. Hence,

$$\Lambda_t D_t^j dt + E_t \left[P_t^j d\Lambda_t + \Lambda_t dP_t^j + d\Lambda_t dP_t^j \right] = 0$$

- Hence,

$$\frac{\Lambda_t D_t^j dt}{\Lambda_t P_t^j} + E_t \left[\frac{P_t^j d\Lambda_t}{\Lambda_t P_t^j} + \frac{\Lambda_t dP_t^j}{\Lambda_t P_t^j} + \frac{d\Lambda_t dP_t^j}{\Lambda_t P_t^j} \right] = 0$$

$$\text{So } \frac{D_t^j}{P_t^j} dt + E_t \left[\frac{d\Lambda_t}{\Lambda_t} + \frac{dP_t^j}{P_t^j} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t^j}{P_t^j} \right] = 0 ...(*)$$

- Summary so far:

	Discrete Time	*Continuous Time*
Price	P_t^j	P_t^j
Dividend	D_{t+1}^j	$D_t^j dt$
Consumer's preferences	$E_t \left[\sum_{\tau=0}^{\infty} \beta^\tau u(C_{t+\tau}) \right]$	$E_t \left[\int_0^\infty e^{-\delta\tau} u(C_{t+\tau}) d\tau \right]$
Fundamental equation	$P_t^j = E_t \left[\sum_{\tau=0}^{\infty} m_{t,t+\tau} D_{t+\tau}^j \right]$	$\Lambda_t P_t^j = E_t \left[\int_0^\infty \Lambda_{t+\tau} D_{t+\tau}^j d\tau \right]$
	where $m_{t,t+\tau} \equiv \beta^\tau \frac{u'(C_{t+\tau})}{u'(C_t)}$	where $\Lambda_{t+\tau} \equiv e^{-\delta(t+\tau)} u'(C_{t+\tau})$
Alternative version	$P_t^j = E_t[m_{t+1}(D_{t+1}^j + P_{t+1}^j)]$	$\Lambda_t D_t^j dt + E_t \left[d(\Lambda_t P_t^j) \right] = 0$
		$\frac{D_t^j}{P_t^j} dt + E_t \left[\frac{d\Lambda_t}{\Lambda_t} + \frac{dP_t^j}{P_t^j} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t^j}{P_t^j} \right] = 0$

- **D. Continuous-Time Fundamental Equation (corresponding to** $1 = E_t[m_{t+1}(1 + r_{t+1}^j)]$**)**

- In discrete time, the net rate of return is

$$\underbrace{r_{t+1}^j}_{\text{net rate of return}} \equiv \underbrace{R_{t+1}^j}_{\text{gross rate of return}} - 1$$

- This is a one-period return between t and $t+1$. To clearly write that it is *from* t,

$$\underbrace{r_{t,t+1}^j}_{\text{net rate of return between } t \text{ and } t+1} \equiv \underbrace{R_{t,t+1}^j}_{\text{gross rate of return between } t \text{ and } t+1} - 1$$

- But $R_{t,t}^j = 1$ for any asset j. That is, if you buy an asset paying \$1 and sell it immediately, then you get \$1. The gross return is simply $R_{t,t}^j = \frac{1}{1} = 1$.

- Hence,

$$r^j_{t,t+1} \equiv R^j_{t,t+1} - R^j_{t,t}$$

So net rate of return, $r_{t,t+1}$, is how the gross return increases by holding the asset for one more period, from t to $t+1$.

- The continuous-time counterpart for $r^j_{t,t+1} \equiv R^j_{t,t+1} - R^j_{t,t}$ is

$$dR^j_t.$$

- In discrete time,

$$\begin{aligned}
\underbrace{r^j_{t+1}}_{\text{net rate of return}} &= \underbrace{R^j_{t+1}}_{\text{gross rate of return}} - 1 \\
&= \frac{P^j_{t+1} + D^j_{t+1} - P^j_t}{P^j_t} \\
&= \frac{P^j_{t+1} - P^j_t}{P^j_t} + \frac{D^j_{t+1}}{P^j_t}
\end{aligned}$$

- The counterpart to the continuous time is

$$dR^j_t = \frac{dP^j_t}{P^j_t} + \frac{D^j_t}{P^j_t} dt$$

This is the return for between t and $t + \Delta$ (or $t + dt$).

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	Discrete Time	*Continuous Time*
Price	P^j_t	P^j_t
Dividend	D^j_{t+1}	$D^j_t dt$
Consumer's preferences	$E_t\left[\sum_{\tau=0}^{\infty} \beta^\tau u(C_{t+\tau})\right]$	$E_t\left[\int_0^\infty e^{-\delta\tau} u(C_{t+\tau}) d\tau\right]$
Fundamental equation	$P^j_t = E_t\left[\sum_{\tau=0}^{\infty} m_{t,t+\tau} D^j_{t+\tau}\right]$	$\Lambda_t P^j_t = E_t\left[\int_0^\infty \Lambda_{t+\tau} D^j_{t+\tau} d\tau\right]$
	where $m_{t,t+\tau} \equiv \beta^\tau \frac{u'(C_{t+\tau})}{u'(C_t)}$	where $\Lambda_{t+\tau} \equiv e^{-\delta(t+\tau)} u'(C_{t+\tau})$
Alternative version	$P^j_t = E_t[m_{t+1}(D^j_{t+1} + P^j_{t+1})]$	$\Lambda_t D^j_t dt + E_t\left[d(\Lambda_t P^j_t)\right] = 0$
		$\frac{D^j_t}{P^j_t} dt + E_t\left[\frac{d\Lambda_t}{\Lambda_t} + \frac{dP^j_t}{P^j_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dP^j_t}{P^j_t}\right] = 0$
Net rate of return	$r^j_{t+1} = \frac{P^j_{t+1} - P^j_t}{P^j_t} + \frac{D^j_{t+1}}{P^j_t}$	$dR^j_t = \frac{dP^j_t}{P^j_t} + \frac{D^j_t}{P^j_t} dt$

- Write (*) as

$$\frac{D_t^j}{P_t^j}dt + E_t\left[\frac{d\Lambda_t}{\Lambda_t} + \frac{dP_t^j}{P_t^j} + \frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j}\right] = 0...(*)$$

$$E_t\left[\underbrace{\frac{dP_t^j}{P_t^j} + \frac{D_t^j}{P_t^j}dt}_{dR_t^j}\right] + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j}\right]$$

- Hence,

$$E_t\left[dR_t^j\right] + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j}\right]...(**)$$

- This is a counterpart to $1 = E_t[m_{t+1}(1+r_{t+1}^j)]$, except that we still have $\frac{dP_t^j}{P_t^j}$.

- What can be done? We will come back after we study $\frac{d\Lambda_t}{\Lambda_t}$ more.

4. Example: CRRA Preferences

- For CRRA:

$$\begin{aligned} u(C_t) &= \frac{C_t^{1-\gamma}}{1-\gamma} \\ \text{So } u'(C_t) &= C_t^{-\gamma} \\ \text{So } \Lambda_{t+\tau} &\equiv e^{-\delta(t+\tau)}u'(C_{t+\tau}) = e^{-\delta(t+\tau)}C_{t+\tau}^{-\gamma} \\ \text{and } \Lambda_t &\equiv e^{-\delta t}C_t^{-\gamma} \end{aligned}$$

- Assume

$$\frac{dC_t}{C_t} = \mu dt + \sigma dz_t$$

(For discrete time, we made a similar assumption for the growth of Y_t. Of course, in market clearing, $Y_t = C_t$, so I will just keep going with C_t.)

- Contents:
  - A. Obtaining $\frac{d\Lambda_t}{\Lambda_t}$
  - B. Revisiting (**)
  - C. Risk-free Rate

– D. Expected Excess Return

- **A. Obtaining $\frac{d\Lambda_t}{\Lambda_t}$**
- We need $\frac{d\Lambda_t}{\Lambda_t}$.
- Problem: What is $d\Lambda_t$ where $\Lambda_t = f(t, C_t) = e^{-\delta t} C_t^{-\gamma}$?
- Ito's lemma:

$$d\Lambda_t = \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} \underbrace{dt^2}_{=0 \text{ by Theorem 2}} + \frac{\partial f}{\partial C_t} dC_t + \frac{1}{2} \frac{\partial^2 f}{\partial C_t^2} dC_t^2 + \frac{\partial^2 f}{\partial t \partial C_t} \underbrace{dt dC_t}_{(A)}$$

- Also,

$$\begin{aligned} (A) &= dt\,(C_t \mu dt + C_t \sigma dz_t) \\ &= C_t \mu \underbrace{dt^2}_{=0 \text{ by Theorem 2}} + C_t \sigma \underbrace{dt dz_t}_{=0 \text{ by Theorem 2}} \\ &= 0 \end{aligned}$$

- So now it is

$$d\Lambda_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial C_t} dC_t + \frac{1}{2} \frac{\partial^2 f}{\partial C_t^2} dC_t^2$$

- Hence,

1. $\frac{\partial f}{\partial t} = -\delta e^{-\delta t} C_t^{-\gamma}$.
2. $\frac{\partial f}{\partial C_t} = -\gamma e^{-\delta t} C_t^{-\gamma-1}$.
3. $\frac{\partial^2 f}{\partial C_t^2} = -\gamma(-\gamma - 1) e^{-\delta t} C_t^{-\gamma-2} = \gamma(\gamma+1) e^{-\delta t} C_t^{-\gamma-2}$.

- Therefore,

$$d\Lambda_t = -\delta e^{-\delta t} C_t^{-\gamma} dt - \gamma e^{-\delta t} C_t^{-\gamma-1} dC_t + \frac{1}{2} \gamma(\gamma+1) e^{-\delta t} C_t^{-\gamma-2} dC_t^2$$

- Hence,

$$\begin{aligned} \frac{d\Lambda_t}{\Lambda_t} &= -\frac{\delta e^{-\delta t} C_t^{-\gamma}}{e^{-\delta t} C_t^{-\gamma}} dt - \frac{\gamma e^{-\delta t} C_t^{-\gamma-1}}{e^{-\delta t} C_t^{-\gamma}} dC_t + \frac{1}{2} \frac{\gamma(\gamma+1) e^{-\delta t} C_t^{-\gamma-2}}{e^{-\delta t} C_t^{-\gamma}} dC_t^2 \\ &= -\delta dt - \gamma \frac{dC_t}{C_t} + \frac{1}{2} \gamma(\gamma+1) \frac{dC_t^2}{C_t^2} \end{aligned}$$

- But

$$\frac{dC_t}{C_t} = \mu dt + \sigma dz_t,$$
$$\left(\frac{dC_t}{C_t}\right)^2 = (\mu dt + \sigma dz_t)^2$$
$$= \mu^2 \underbrace{dt^2}_{=0 \text{ by Theorem 2}} + \sigma^2 \underbrace{dz_t^2}_{=dt \text{ by Theorem 1}} + 2\mu\sigma \underbrace{dt dz_t}_{=0 \text{ by Theorem 2}}$$
$$= \sigma^2 dt$$

- Hence,

$$\frac{d\Lambda_t}{\Lambda_t} = -\delta dt - \gamma(\mu dt + \sigma dz_t) + \frac{1}{2}\gamma(\gamma+1)\sigma^2 dt$$
$$= -\left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right] dt - \gamma\sigma dz_t$$

- **B. Revisiting** $(**)$
- Let's consider this first:

$$\frac{d\Lambda_t}{\Lambda_t} dR_t^j = \frac{d\Lambda_t}{\Lambda_t}\left(\frac{dP_t^j}{P_t^j} + \frac{D_t^j}{P_t^j} dt\right)$$
$$= \frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j} + \frac{d\Lambda_t}{\Lambda_t}\frac{D_t^j}{P_t^j} dt$$
$$= \frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j} + \underbrace{\left(-\left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right] dt - \gamma\sigma dz_t\right)\frac{D_t^j}{P_t^j} dt}_{=0 \text{ since } dt^2=0 \text{ and } dz_t dt=0}$$
$$= \frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j}$$

- So $(**)$ becomes

$$E_t\left[dR_t^j\right] + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t^j}{P_t^j}\right] ...(**)$$
$$= -E_t\left[\frac{d\Lambda_t}{\Lambda_t} dR_t^j\right] ...(***)$$

- Now we have eliminated $\frac{dP_t^j}{P_t^j}$ in the fundamental equation.

- This is a counterpart to $1 = E_t[m_{t+1}(1 + r^j_{t+1})]$. Done!
- **C. Risk-free Rate**
- What is

$$dR^f_t?$$

- Risk-free asset is special. You know this rate at t.
- If an investor knows the risk-free rate from t will be r^f_t, then

$$dR^f_t = r^f_t dt.$$

- That is, for one period, it pays r^f_t. So for an infinitesmal period dt, it pays $r^f_t dt$.
- Inserting $dR^f_t = r^f_t dt$ into $(***)$,

$$E_t\left[r^f_t dt\right] + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} r^f_t dt\right].$$

- But
  - $E_t\left[r^f_t dt\right] = r^f_t dt$ since r^f_t is risk-free.
  - $E_t\left[\frac{d\Lambda_t}{\Lambda_t} r^f_t dt\right] = E_t\left[\left(-\left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right] dt - \gamma\sigma dz_t\right) r^f_t dt\right] = 0$ since $dt^2 = 0$ and $dz_t dt = 0$.
- Hence,

$$r^f_t dt + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = 0...(****)$$

- But

$$\begin{aligned} E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] &= E_t\left[-\left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right] dt - \gamma\sigma dz_t\right] \\ &= -\left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right] dt \text{ (since } E_t[dz_t] = 0) \end{aligned}$$

- Finally,

$$r^f_t dt = \left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right] dt.$$

- **Interpretation**: The risk-free rate, r^f_t, is higher when

1. δ (impatience) is higher
2. μ (average consumption growth) is higher: The investor expects higher consumption in the future anyway, so the risk-free asset becomes less popular. This implies that the risk-free rate should be higher to keep it attractive.
3. σ (volatility of consumption growth) is lower: The investor faces more uncertainty. So risk-free asset is more attractive. Lower risk-free rate is enough for market clearing.

- **D. Asset j's Expected Excess Return**
- Plugging $(****)$ to $(***)$,

$$E_t\left[dR_t^j\right] + \underbrace{E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right]}_{-r_t^f dt} = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} dR_t^j\right] ...(***)$$

$$\text{So } \underbrace{E_t\left[dR_t^j\right] - r_t^f dt}_{\text{equity premium}} = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} dR_t^j\right]$$

- But

$$\begin{aligned} cov_t\left[\frac{d\Lambda_t}{\Lambda_t}, dR_t^j\right] &= E_t\left[\frac{d\Lambda_t}{\Lambda_t} dR_t^j\right] - E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] E_t\left[dR_t^j\right] \\ &= E_t\left[\frac{d\Lambda_t}{\Lambda_t} dR_t^j\right] \end{aligned}$$

- Trick: This is because both $\frac{d\Lambda_t}{\Lambda_t}$ and dR_t^j are contiunous-time stochastic process, in the form of ______$dt +$ ______dz_t. That is, both $E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right]$ and $E_t\left[dR_t^j\right]$ are in the form of ______dt, so $E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] E_t\left[dR_t^j\right] = 0$ since $dt^2 = 0$.

- Hence,

$$\begin{aligned}
\underbrace{E_t\left[dR_t^j\right] - r_t^f dt}_{\text{expected excess return}} &= -cov_t\left[\frac{d\Lambda_t}{\Lambda_t}, dR_t^j\right] \\
&= -cov_t\left[\underbrace{-\left[\delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2\right]dt}_{\text{constant}} - \gamma\sigma dz_t, dR_t^j\right] \\
&= -cov_t\left[\underbrace{-\gamma\sigma}_{\text{const}} dz_t, dR_t^j\right] \\
&= -(-\gamma\sigma)cov_t\left[dz_t, dR_t^j\right] \\
&= \gamma\sigma cov_t\left[dz_t, dR_t^j\right]
\end{aligned}$$

- **Interpretation**: Assuming $cov_t\left[dz_t, dR_t^j\right]$ is positive (assuming j's return is positively correlated with consumption shock), the expected excess return on asset j is higher if

  1. γ (risk aversion) is higher (so this asset is less attractive for consumption insurance)
  2. σ (volatility of consumption growth) is higher

- Formula: For $\Lambda_t = f(t, C_t)$,

$$d\Lambda_t = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial C_t}dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial C_t^2}dC_t^2 + \frac{\partial^2 f}{\partial t \partial C_t}dtdC_t.$$

1. Consider the following fundamental equation of asset pricing in continuous time:

$$E_t\left[dR_t^j\right] + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t}dR_t^j\right],$$

where dR_t^j is the net return on asset j from period t. Also, $\Lambda_t \equiv e^{-\delta t}u'(C_t)$, where δ is a parameter representing impatience, and $u()$, the utility function, satisfies

$$u(C_t) = \log(C_t).$$

Hence, $u'(C_t) = 1/C_t$. In addition, C_t is the consumption, following

$$\frac{dC_t}{C_t} = \mu dt + \sigma dz_t,$$

where μ and σ are constant, and dz_t is an increment for a Brownian motion, satisfying $dz_t \tilde{} N(0, dt)$.

(a) Fill the blanks:

$$\frac{d\Lambda_t}{\Lambda_t} = \boxed{\quad (A) \quad} dt + \boxed{\quad (B) \quad} dz_t,$$

only with parameters, δ, μ, and σ.

Hint: For $\Lambda_t = f(t, C_t)$,

$$d\Lambda_t = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial C_t}dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial C_t^2}dC_t^2 + \frac{\partial^2 f}{\partial t \partial C_t}dtdC_t.$$

Answer: *For* $\Lambda_t = f(t, C_t) = \frac{e^{-\delta t}}{C_t}$,

$$\begin{aligned} d\Lambda_t &= \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial C_t}dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial C_t^2}dC_t^2 + \frac{\partial^2 f}{\partial t \partial C_t}dtdC_t \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial C_t}dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial C_t^2}dC_t^2 \end{aligned}$$

Also,

$$\frac{\partial f}{\partial t} = -\delta \frac{e^{-\delta t}}{C_t},$$
$$\frac{\partial f}{\partial C_t} = -e^{-\delta t} C_t^{-2},$$
$$\frac{\partial^2 f}{\partial C_t^2} = 2e^{-\delta t} C_t^{-3},$$
$$\frac{dC_t}{C_t} = \mu dt + \sigma dz_t,$$
$$\left(\frac{dC_t}{C_t}\right)^2 = (\mu dt + \sigma dz_t)^2 = \sigma^2 dt.$$

Hence,

$$\begin{aligned}
\frac{d\Lambda_t}{\Lambda_t} &= -\frac{\delta \frac{e^{-\delta t}}{C_t}}{\frac{e^{-\delta t}}{C_t}} dt - \frac{e^{-\delta t} C_t^{-2}}{\frac{e^{-\delta t}}{C_t}} dC_t + \frac{1}{2} 2 \frac{e^{-\delta t} C_t^{-3}}{\frac{e^{-\delta t}}{C_t}} dC_t^2 \\
&= -\delta dt - \frac{dC_t}{C_t} + \frac{dC_t^2}{C_t^2} \\
&= -\delta dt - \mu dt - \sigma dz_t + \sigma^2 dt \\
&= [\sigma^2 - \delta - \mu] dt - \sigma dz_t
\end{aligned}$$

(b) Fill the blanks for a risk-free rate, r_t^f:

$$r_t^f dt = \boxed{\text{(C)}}\, dt + \boxed{\text{(D)}}\, dz_t,$$

only with parameters, δ, μ, and σ.

Answer: *Inserting $dR_t^f = r_t^f dt$ into the fundamental equation stated in the problem,*

$$E_t\left[r_t^f dt\right] + E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} r_t^f dt\right].$$

But $E_t\left[r_t^f dt\right] = r_t^f dt$ since r_t^f is risk-free. Also, $E_t\left[\frac{d\Lambda_t}{\Lambda_t} r_t^f dt\right] = 0$. Hence,

$$r_t^f dt + E_t\left[[\sigma^2 - \delta - \mu] dt - \sigma dz_t\right] = 0.$$

$$\text{So } r_t^f dt = [\delta + \mu - \sigma^2] dt.$$

(c) Assume $\delta = 0.05$, $\mu = 0.02$ (based on average annual U.S. GDP growth in data), $\sigma = 0.01$ (so $\sigma^2 = 0.0001$, based on the volatility of annual U.S. GDP growth in data). Discuss the risk-free rate puzzle using your results in (b).

Answer: $r_t^f dt = [0.05 + 0.02 - 0.0001]dt \approx 0.07dt$. *The model's prediction on the risk-free rate is 7%. However, the interest rate on the U.S. government bond is about 0.5% on average since 1946. This failure of the theory to explain the risk-free rate is the risk-free rate puzzle.*

(d) What is the volatility of the risk-free rate implied by this model, given the parameter values in (c)? In one of the problem set questions, you obtained the volatility of the risk-free rate in a similar model. Why are the implied volailities different between the current model and the model in the problem set? Discuss in plain English.

Answer: *The risk-free rate in (b) is constant (i.e., there is no dz_t). Hence, the volatility is also zero.*

In one of the past problem set questions, the implied volaility was positive, at about 3%. A difference is that this model assumes log utility (so $\gamma = 1$) while the previous problem set assumed a higher level of risk aversion. More important, in this problem, the consumption process is i.i.d. over time, so every period has no new information about the next period's consumption growth, while in the problem set question, there was a negative autocorrelation of aggregate shocks, so today's economy has some information about the future economy, and hence, the risk-free rate changes reflecting that information. This creates the volatility of risk-free rate.

Inflation Risk: Cash-In-Advance Model

References

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Contents

1. Deterministic Control Problem

- This part discusses the last technique in EconS 502: Bellman equation.
- **Example**:

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u \left(\underbrace{f(K_t) - K_{t+1}}_{=C_t} \right)$$

  K_0 is given, where $f(K_t)$ is the production given the physical capital stock, K_t, and K_{t+1} is the savings (which become the next period's physical capital stock).

- A usual way to solve it:

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} u\left(f(K_0)-K_1\right)+\beta u\left(f(K_1)-K_2\right)+...+\beta^t u\left(f(K_t)-K_{t+1}\right)+\beta^{t+1} u\left(f(K_{t+1})-K_{t+2}\right)+...$$

The FOC wrt K_{t+1} is

$$-\beta^t u'\left(f(K_t)-K_{t+1}\right)+\beta^{t+1} u'\left(f(K_{t+1})-K_{t+2}\right) f'(K_{t+1})=0$$

$$\text{So } u'\left(f(K_t)-K_{t+1}\right)=\beta u'\left(f(K_{t+1})-K_{t+2}\right) f'(K_{t+1})$$

- The problem can be alternatively written as

$$V(K)=\max_{K'}[u(f(K)-K')+\beta V(K')]$$

where $V(K)$ is "all future utilities discounted to today", given this period's physical capital stock, K.

- K' is the next period's. (The next period is usually denoted by "$'$".)

- **Bellman Equation**

  - Problem: $V(x)=\max_y F(x,y)+\beta V(x')$ where $x'=g(x,y)$.
  - First, make it unconstrained: $V(x)=\max_y F(x,y)+\beta V(g(x,y))$
    $V()$: "value function"
    x: "state variable" (affected by the choice of y)
    y: "control variable"
    x': state variable in the next period
  - **A. FOC**: Differentiate wrt control variable (y) and set it equal to 0:

$$\boxed{F_2(x,y)+\beta\frac{\partial V(g(x,y))}{\partial y}=0}$$

  - A subscript "2" in $F_2(x,y)$ means the derivative wrt the second variable, y.
  - It can be alternatively written as

$$F_2(x,y)+\beta V'(g(x,y))g_2(x,y)=0$$

  - **B. Envelope Condition**: Differentiate the whole equation wrt state variable (x).

$$\boxed{V\prime(x)=F_1(x,y)+\beta\frac{\partial V(g(x,y))}{\partial x}}$$

    It can be alternatively written as

$$V'(x)=F_1(x,y)+\beta V'(g(x,y))g_1(x,y)$$

- Now go back to the example.
- Control: K'. State: K.

$$\text{FOC: } -u'(f(K)-K')+\beta V'(K')=0$$
$$\text{EC: } V'(K)=u'(f(K)-K')f'(K)$$

- Now $V'(K)=u'(f(K)-K')f'(K)$ implies $V'(K')=u'(f(K')-K'')f'(K')$. Plugging this to the FOC,

$$u'(f(K)-K')=\beta u'(f(K')-K'')f'(K')$$

So we have the same solutions.

2. Stochastic Control Problem

- We now deal with more sophisticated set-ups. Assume s_t is first-order Markov.
- **Bellman Equation** (Ljungqvist and Sargent, 2nd Edition, p.92.)
  - Problem: $V(x,s)=\max_y F(x,y,s)+\beta E[V(\underbrace{g(x,y,s)}_{=x'},s')|s]$

    s: state variable - exogenous shock

    x: state variable - "endogenous" (affected by the choice of y)

    y: control variable
  - $\boxed{\text{FOC: } F_2(x,y,s)+\beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial y}|s\right]=0}$ (Differentiate wrt control (y)!)
  - $\boxed{\text{Envelope Condition: } V_1(x,s)=F_1(x,y,s)+\beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial x}|s\right]}$ (Differentiate wrt state (x)!)
- If there are constraints, add them just as in Lagrangian:
  - Problem: $V(x,s)=\max_y F(x,y,s)+\beta E[V(\underbrace{g(x,y,s)}_{=x'},s')|s]$

    s.t. $G(x,y,s)\geq 0$
  - $\boxed{\text{FOC: } F_2(x,y,s)+\beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial y}|s\right]+\underbrace{\lambda(s)G_2(x,y,s)}_{\text{new!}}=0}$ (Differentiate wrt control (y)!)

– $\boxed{\text{Envelope Condition: } V_1(x,s) = F_1(x,y,s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial x}|s\right] + \underbrace{\lambda(s)G_1(x,y,s)}_{\text{new!}}}$

(Differentiate wrt state (x)!)

- In all future exams (including preliminary exams) in academic year 2010-11, the exam problems written by S. Choi will provide the following formulae (or their reasonable variations) if they are needed to solve the problems:

1. Hamiltonian: For $\max_{\{u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t))dt$ s.t. $\dot{x}(t) = g(x(t), u(t))$, $x(0)$ given,

$$H(x(t), u(t), \lambda(t)) = h(x(t), u(t)) + \lambda(t)g(x(t), u(t)),$$

and the first-order conditions are $H_u = 0$ and $H_x = \rho\lambda(t) - \dot{\lambda}(t)$.

2. Ito's Lemma: For $\Lambda_t = f(t, C_t)$,

$$d\Lambda_t = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial C_t}dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial C_t^2}dC_t^2 + \frac{\partial^2 f}{\partial t \partial C_t}dtdC_t.$$

3. Bellman Equation: $V(x,s) = \max_y F(x,y,s) + \beta E[V(g(x,y,s),s')|s]$ s.t. $G(x,y,s) \geq 0$ (where $x' = g(x,y,s)$)
   - First-order Condition: $F_y(x,y,s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial y}|s\right] + \lambda(s)G_y(x,y,s) = 0$
   - Envelope Condition: $V_x(x,s) = F_x(x,y,s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial x}|s\right] + \lambda(s)G_x(x,y,s)$

- This policy applies to the exam problems by S. Choi only. If this policy changes (which I do not believe will happen), S. Choi will announce in a reasonabe time before the exam to which it applies.

3. Cash-In-Advance Model

- Growth: Long-run trend of output

  Asset Pricing: "Uncertainty" added

  Here: "Money" added

- Endowment economy, Representative consumer

- **A. Endowment and Money Supply**

- Y_t: (stochastic, exogenous) endowment

  No storage $\Longrightarrow$ In equilibrium, $C_t = Y_t$

- $\overline{M}_t$: (stochastic, exogenous) money supply

  $\Longrightarrow \omega_t$: (stochastic, exogenous) gross money growth rate, satisfying $\overline{M}_t = \omega_t \overline{M}_{t-1}$

  (injection of new money: $(\omega_t - 1)\overline{M}_{t-1}$. New money is lump-sum transferred to the consumer.)

- How to describe an evolution of stochastic, exogenous variables:

  $s_t = (Y_t, \omega_t)$: **Macroeconomic shock**. Assume s_t is first-order Markov.

  $F(\underbrace{s'}_{\text{next period}} : \underbrace{s}_{\text{this period}})$ is conditional distribution.

- **B. Asset Market (Bond Market)**

- Q_t: \$-price of one-period **nominal** bond

  +————————————+

  time t — time $t+1$

  Pay \$$Q_t$ — Receive \$1

- N_t: Units (#) of nominal bonds held by the Representative consumer between t and $t+1$

- In equilibrium, Net demand$= 0 \Longrightarrow N_t = 0$

- **C. Goods Market**

- P_t: \$-price of one unit of physical good

- In equilibrium, $Y_t = C_t$

- **D. The Problem**

- A representative consumer maximizes the expected value of the present value of

$$u(C_t)$$

  for all current and future period, t.

- $t-1$ t

+————————+————————+

↑

With whatever amount of money she has:

(i) Buy N_{t-1} units of bonds (paying $\$Q_{t-1}N_{t-1}$)

(ii) Hold the remaining as $\$M_{t-1}$

With Y_{t-1} units of physical goods,

(i) Eat C_{t-1}

(ii) Sell $Y_{t-1}-C_{t-1}$ for additional \$, at price P_{t-1}

↑

$\$M_{t-1}$ in the pocket

$\$N_{t-1}$ received from the investment

$\$P_{t-1}(Y_{t-1}-C_{t-1})$

$\$(\omega_t-1)\overline{M}_{t-1}$ transacted by the central bank

With this money,

(i) Buy N_t units of bonds (paying $\$Q_tN_t$)

(ii) Hold the remaining as $\$M_t$

With Y_t units of physical goods,

(i) Eat C_t

(ii) Sell Y_t-C_t for additional \$, at price P_t

- Constraint 1: **Budget constraint**:

$$(\text{Use of \$}) \leq (\text{budget in \$})$$

$$M_t+Q_tN_t \leq M_{t-1}+N_{t-1}+P_{t-1}(Y_{t-1}-C_{t-1})+(\omega_t-1)\overline{M}_{t-1}$$

for all t.

- Constraint 2: **Cash-In-Advance (CIA) constraint**: Need cash to consume (even though you consume your own endowment)!

$$P_tC_t \leq M_t,$$

for all t.

- This is an "artificial" constraint in some sense. This does not explain why we need money, but assumes that we need money to consume.
- In **equilibrium**, the market clears:

$$\begin{aligned} N_t &= 0 \\ Y_t &= C_t \end{aligned}$$

- Hence,

$$M_t + \underbrace{Q_t N_t}_{=0} = M_{t-1} + \underbrace{N_{t-1}}_{=0} + \underbrace{P_{t-1}(Y_{t-1} - C_{t-1})}_{=0} + (\omega_t - 1)\overline{M}_{t-1}$$

- Hence,

$$M_t - M_{t-1} = \overline{M}_t - \overline{M}_{t-1}.$$

- Assuming $M_0 = \overline{M}_0$ for the initial period 0, we have

$$\left.\begin{aligned} N_t &= 0 \\ Y_t &= C_t \end{aligned}\right\} \Longrightarrow \underbrace{M_t}_{\text{money demand}} = \underbrace{\overline{M}_t}_{\text{money supply}}$$

  for all t.

4. Bellman Equation for Cash-In-Advance Model

- Further assumption: We look for an equilibrium where

$$\begin{aligned} P_t &= \overline{M}_t p(s_t)\text{: Goods price will be proportional to M} \\ Q_t &= q(s_t) \text{ : Bond price will not depend on M} \end{aligned}$$

  (Reasonable!)

- Constraint 1 - **Budget constraint**:

$$\begin{aligned} M_t + Q_t N_t &\leq M_{t-1} + N_{t-1} + P_{t-1}(Y_{t-1} - C_{t-1}) + (\omega_t - 1)\overline{M}_{t-1}. \\ \text{So } M_t + q(s_t) N_t &\leq M_{t-1} + N_{t-1} + \overline{M}_{t-1} p(s_{t-1})(Y_{t-1} - C_{t-1}) + (\omega_t - 1)\overline{M}_{t-1}. \end{aligned}$$

  Normalize by $\overline{M}_t$.

$$\text{So } \frac{M_t}{\overline{M}_t} + q(s_t)\frac{N_t}{\overline{M}_t} \leq \frac{\overline{M}_{t-1}}{\overline{M}_t}\left[\frac{M_{t-1}}{\overline{M}_{t-1}} + \frac{N_{t-1}}{\overline{M}_{t-1}} + p(s_{t-1})(Y_{t-1} - C_{t-1}) + (\omega_t - 1)\right].$$

- Define $m_t = \frac{M_t}{\overline{M}_t}$ and $n_t = \frac{N_t}{\overline{M}_t}$.

$$m_t + q(s_t)n_t \leq \frac{1}{\omega_t} \underbrace{[m_{t-1} + n_{t-1} + p(s_{t-1})(Y_{t-1} - C_{t-1}) + (\omega_t - 1)]}_{\equiv \alpha_t \text{ (total asset at the beginning of } t)}$$

- Constraint 2 - **CIA constraint**:

$$\begin{aligned} P_t C_t &\leq M_t \\ \text{So } \overline{M}_t p(s_t) C_t &\leq M_t \\ \text{So } p(s_t) C_t &\leq m_t \end{aligned}$$

- We can write the **Bellman equation** as follows.

$$V(\alpha, s) = \max_{C,m,n} \left[u(C) + \beta E \left[V(\alpha', s') | s \right] \right]$$

subject to

$$\begin{aligned} p(s)C &\leq m, \\ m + q(s)n &\leq \alpha/\omega \\ \alpha' &= m + n + p(s)(Y - C) + \omega' - 1 \end{aligned}$$

- Definition: A **recursive competitive equilibrium** is $\{C(s), m(s), n(s)\}$ and $\{p(s), q(s)\}$ s.t.

  (i) $\{C(s), m(s), n(s)\}$ solve the consumer's problem given $\{p(s), q(s)\}$, and

  (ii) Market clears, i.e, $m(s) = 1$, $n(s) = 0$, $C(s) = Y$.

- Eliminating α',

$$V(\alpha, s) = \max_{C,m,n} \left[u(C) + \beta E[V(\underbrace{m + n + p(s)(Y - C) + \omega' - 1}_{=\alpha'}, s')|s] \right]$$

s.t.

$$\begin{aligned} m - p(s)C &\geq 0, \quad [\mu(s)] \\ a/\omega - m - q(s)n &\geq 0. \quad [\lambda(s)] \end{aligned}$$

- The FOC wrt C:

$$u'(C) - \beta \underbrace{E[V_1(\alpha', s')p(s)|s]}_{=p(s)E[V_1(\alpha',s')|s]} - \mu(s)p(s) = 0.$$

So

$$(1) \ u'(C) = p(s) \left[\mu(s) + \beta E[V_1(\alpha', s')|s] \right].$$

- The FOC wrt m:
$$\beta E[V_1(\alpha', s')|s] + \mu(s) - \lambda(s) = 0.$$
So
$$(2)\ \lambda(s) = \mu(s) + \beta E[V_1(\alpha', s')|s]$$
- The FOC wrt n:
$$(3)\ \beta E[V_1(\alpha', s')|s] - \lambda(s)q(s) = 0.$$
- The EC:
$$(4)\ V_1(\alpha, s) = \lambda(s)/\omega.$$
- <Step 1> Eliminate $V_1(\alpha, s)$ using
$$\begin{aligned} (4)\text{: } & V_1(\alpha, s) = \lambda(s)/\omega. \\ \text{So: } & V_1(\alpha', s') = \lambda(s')/\omega' \end{aligned}$$
$$\begin{aligned} (1)\text{: } & u'(C) = p(s)\left[\mu(s) + \beta E[\lambda(s')/\omega'|s]\right]. \\ (2)\text{: } & \lambda(s) = \mu(s) + \beta E[\lambda(s')/\omega'|s] \\ (3)\text{: } & \beta E[\lambda(s')/\omega'|s] - \lambda(s)q(s) = 0 \end{aligned}$$
- <Step 2> Eliminate $\mu(s)$ using (2):
$$\begin{aligned} (1)\text{: } & u'(C) = p(s)\lambda(s). \\ (3)\text{: } & \beta E[\lambda(s')/\omega'|s] - \lambda(s)q(s) = 0 \text{ (same)} \end{aligned}$$
- <Step 3> Eliminate $\lambda(s)$ using
$$\begin{aligned} (1)\text{: } & \lambda(s) = u'(C)/p(s). \\ \text{So: } & \lambda(s') = u'(C')/p(s'). \end{aligned}$$
$$(3)\text{: } \beta E\left[\left.\frac{u'(C')}{\omega' p(s')}\right| s\right] - \frac{u'(C)}{p(s)}q(s) = 0.$$
- Rearranging,
$$q(s) = E\left[\left.\beta\frac{u'(C')p(s)}{u'(C)\omega' p(s')}\right| s\right]$$
- Compare: Fundamental eqaution in discrete time:
$$\tilde{P}_t^j = E_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}X_{t+1}^j\right]$$
where $\tilde{P}_t^j$ is the price of this asset.

- But this is in units of REAL goods. Not in NOMINAL dollars:

	t	$t+1$
Fundamental Eq.	Pay $\tilde{P}_t^j$ units	Receive X_{t+1}^j units
This Model	Pay $\$q(s_t)$	Receive $1
	(But 1 unit=$\$P_t = \$\overline{M}_t p(s_t)$)	(1 unit=$\$P_{t+1} = \$\overline{M}_{t+1} p(s_{t+1})$)
	(So $\$1 = \frac{1}{\overline{M}_t p(s_t)}$ units)	($\$1 = \frac{1}{\overline{M}_{t+1} p(s_{t+1})}$ units)
	(So $\$q(s) = \frac{q(s_t)}{\overline{M}_t p(s_t)}$ units)	
	So replace $\tilde{P}_t^j$ by $\frac{q(s_t)}{\overline{M}_t p(s_t)}$	So replace X_{t+1}^j by $\frac{1}{\overline{M}_{t+1} p(s_{t+1})}$

- Then:

$$\frac{q(s_t)}{\overline{M}_t p(s_t)} = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{\overline{M}_{t+1} p(s_{t+1})}\right]$$

$$\text{So } q(s_t) = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{\overline{M}_t p(s_t)}{\overline{M}_{t+1} p(s_{t+1})}\right]$$

$$= E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{p(s_t)}{\omega_{t+1} p(s_{t+1})}\right]$$

So what we drived is just a special case of the (old) fundamental eq.

- Equilibrium: First, $Y = C$.

$$\text{(A) } q(s) = E\left[\beta \frac{u'(C')p(s)}{u'(C)\omega' p(s')}\middle| s\right]$$

$$= E\left[\beta \frac{u'(Y')p(s)}{u'(Y)\omega' p(s')}\middle| s\right]$$

- Also, assume the CIA constraint is binding (so the consumer holds the minimum possible dollars so that she can invest everything else in bonds), so

$$p(s)C = m.$$

In equilibrium, $Y = C$ and $m = 1$. So

$$\text{(B) } p(s) = \frac{1}{Y}.$$

- You might want to plug (B) into (A):

$$\text{(A') } q(s) = E\left[\beta \frac{u'(Y')/Y}{u'(Y)\omega'/Y'}\middle| s\right]$$

$$= E\left[\beta \frac{Y'u'(Y')}{Yu'(Y)\omega'}\middle| s\right]$$

5. Implications on Nominal Interest Rates

- **Example 1**: $Y_t = \overline{Y}$ (constant endowment), $\omega_t = \overline{\omega}$ (constant money growth).
- (B) becomes

$$p(s) = \frac{1}{Y} = \frac{1}{\overline{Y}} \text{ (constant)}$$

- (A) becomes

$$\begin{aligned} q(s) &= E\left[\beta \frac{u'(Y')p(s)}{u'(Y)\omega' p(s')}\middle| s\right] \\ &= E\left[\beta \frac{1}{\omega'}\middle| s\right] \text{ (}Y\text{ and }p(s)\text{ constant)} \\ &= \frac{\beta}{\overline{\omega}} \end{aligned}$$

- Bond price is constant.
- Nominal interest rate: $1 + i(s) = \frac{\$1}{\$q(s)} = \frac{\overline{\omega}}{\beta}$ (depending on money growth rate and time preference.)
- **Example 2**: $Y_t = \overline{Y}$ (constant endowment), ω_t i.i.d.
- Implication on (B) is the same.
- (A) becomes

$$\begin{aligned} q(s) &= E\left[\beta \frac{u'(Y')p(s)}{u'(Y)\omega' p(s')}\middle| s\right] \\ &= E\left[\beta \frac{1}{\omega'}\middle| s\right] \text{ (}Y\text{ and }p(s)\text{ constant)} \\ &= \beta \underbrace{E\left[\frac{1}{\omega'}\middle| s\right]}_{\text{const since }\omega\text{ is i.i.d.}} \end{aligned}$$

- Bond price is still constant. But the level depends on the average of (the inverse of) money growth.
- Nominal interest rate is constant.
- **Example 3**: $Y_t = \overline{Y}$ (constant endowment), ω_t first-order Markov. Especially, if ω is high, ω' is more likely to be high.

- (A) becomes

$$q(s) = \beta \underbrace{E\left[\left.\frac{1}{\omega'}\right| s\right]}_{\text{no longer constant}}.$$

- So bond price varies over time.

- Case 1: Current ω is high, so ω' is also expected to be high. Then $E\left[\left.\frac{1}{\omega'}\right| s\right]$ is low. $q(s)$ is low. $1 + i(s) = \frac{\$1}{\$q(s)}$ is high. **Higher money growth rate today raises nominal interest rate immediately.**

- **Example 4**: Y_t first-order Markov, ω_t first-order Markov. Especially, if ω is high, ω' is more likely to be high.

- Recall (B):

$$\text{(B) } p(s) = \frac{1}{Y},$$
$$p(s') = \frac{1}{Y'}.$$

- (A):

$$q(s) = E\left[\left.\beta \frac{u'(Y')p(s)}{u'(Y)\omega' p(s')}\right| s\right].$$

- As usual, assume

$$u(C) = \frac{C^{1-\theta}}{1-\theta}, \ \theta > 0 \text{ and } \theta \neq 0$$

and $u(C) = \log C$ as $\theta \longrightarrow 0$. Then,

$$\begin{aligned} q(s) &= E\left[\left.\beta \left(\frac{Y'}{Y}\right)^{-\theta} \frac{1}{\omega'} \left(\frac{Y'}{Y}\right)\right| s\right] \\ &= E\left[\left.\beta \left(\frac{Y'}{Y}\right)^{1-\theta} \frac{1}{\omega'}\right| s\right] \end{aligned}$$

- **Nominal interest rate depends on the current expectation on future GDP growth rate and future money growth.**

6. Extensions

- **Model 1**: Production in Cash-in-Advance Economy

- See Problem Set.
- **Friedman Rule**: The optimal monetary policy is when $i = 0$. Otherwise the economy is losing something at all time.
- **Model 2**: Shopping-Time Model (Transaction-Cost Model)
- θ: Shopping time.

$$V(\alpha, s) = \max_{C,\theta,m,n} \left[u(C) + \beta \int V(\alpha', s') dF(s'|s) \right]$$

subject to

$$p(s)C \leq \underbrace{H(m, \theta)}_{\text{replacing } m \text{ in the original model}} , \text{ increasing in both elements}$$

(If θ is high, m doesn't have to be high..)

$$Y = \widetilde{Y}(\underbrace{1 - \theta}_{\text{amount you work}})$$

$$m + q(s)n \leq \alpha/\omega$$

$$\alpha' = m + n + p(s)(Y - C) + \omega' - 1$$

- **Model 3**: Money-In-Utility Model
- Utility function: $u(C, M)$

EXERCISES

1. Consider an economy with production, where labor is an input and labor productivity is stochastic. To be specific, the representative consumer has preferences

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\theta}}{1-\theta}-H_t\right)\right],$$

for $0<\beta<1$, $\theta>0$ and $\theta\neq 1$, where C_t is consumption (in units of physical goods) and H_t is labor supply (e.g., hours worked). The production is

$$Y_t = z_t H_t \text{ for all } t=0,1,...,$$

where Y_t is production (in units of physical goods) and z_t is stochastic and exogenously determined labor productivity. This is a closed economy, and there is no storage, so $Y_t = C_t$ for all $t=0,1,...,$ in equilibrium.

In this problem, you will study how money growth (or monetary policy) can "distort" the economy. You will first solve the representative consumer's optimization problem without money, and then analyze the same problem with a cash-in-advance constraint.

(a) Suppose that money does not exist. That is, the consumer can consume her own output without any cash holding. At each period, the consumer solves the following problem, given a realized value of z:

$$\max_{C,H}\frac{C^{1-\theta}}{1-\theta}-H$$

subject to $C=zH$. Provide the solution. (This solution is Pareto optimal because the consumer's choice is not altered by the cash-in-advance constraint.)

Answer: *FOC:*

$$z^{1-\theta}H^{-\theta}=1.$$

So $H^*=z^{(1-\theta)/\theta}$ *and* $C^*=z^{1/\theta}$.

(b) Now consider a cash-in-advance economy. The following is a sequence of events at each period t:

1. The productivity shock z_t is realized and the central bank injects (or withdraws) money. Denote by ω_t-1 the net growth rate of money supply (i.e., $\omega_t=\overline{M}_t/\overline{M}_{t-1}$).

2. The consumer allocates her asset between cash holding, M_t, and the holding of (nominal) bond, N_t, at price Q_t. To be specific, she pays $Q_t N_t$ dollars to purchase N_t units of bonds, and at the next period, N_t dollars are delivered to her.

3. Labor is supplied, production occurs, and consumption is determined. Here, the cash-in-advance constraint, $P_t C_t \leq M_t$, should be satisfied.

The consumer's asset at $t+1$ consists of cash and maturing bonds, plus net receipts from the sale of physical goods at the end of t, plus an injection of cash at the beginning of $t+1$. This implies that the budget constraint at $t+1$ becomes

$$M_{t+1} + Q_{t+1} N_{t+1} \leq M_t + N_t + P_t(z_t H_t - C_t) + (\omega_{t+1} - 1)\overline{M}_t.$$

The state vector for the economy, $s = (z, \omega)$, is a first-order Markov process with a transition function $F(s'|s)$ for all s. We look at equilibria in which $P_t = \overline{M}_t p(s_t)$ and $Q_t = q_b(s_t)$. Define $m_t \equiv \frac{M_t}{\overline{M}_t}$ and $n_t \equiv \frac{N_t}{\overline{M}_t}$. The Bellman equation is then

$$V(\alpha, s) = \max_{C,H,m,n} \left[\frac{C^{1-\theta}}{1-\theta} - H + \beta E\left[V(\alpha', s')|s\right] \right]$$

s.t.

$$\begin{aligned} p(s)C &\leq m, \\ m + q_b(s)n &\leq \alpha/\omega \\ \alpha' &= m + n + p(s)(zH - C) + \omega' - 1 \end{aligned}$$

Obtain the first-order conditions and envelope condition.

Hint: Problem: $V(x, s) = \max_y F(x, y, s) + \beta E[V(g(x, y, s), s')|s]$ s.t. $G(x, y, s) \geq 0$ and $x' = g(x, y, s)$.

- First-order Condition: $F_2(x, y, s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial y}|s\right] + \lambda(s) G_2(x, y, s) = 0$

- Envelope Condition: $V_1(x, s) = F_1(x, y, s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial x}|s\right] + \lambda(s) G_1(x, y, s)$

Answer: *Eliminating* α',

$$V(\alpha, s) = \max_{C,H,m,n} \left[\frac{C^{1-\theta}}{1-\theta} - H + \beta E[V(\underbrace{m + n + p(s)(zH - C) + \omega' - 1}_{=\alpha'}, s')|s] \right]$$

s.t.

$$\begin{aligned} m - p(s)C &\geq 0, \quad [\mu(s)] \\ a/\omega - m - q_b(s)n &\geq 0. \quad [\lambda(s)] \end{aligned}$$

The FOC wrt C:

$$C^{-\theta} - \beta p(s) E[V_1(\alpha', s')|s] - \mu(s)p(s) = 0.$$

Note: V_1: Derivative wrt the first element. So

$$(1a)\ C^{-\theta} = p(s)\left[\mu(s) + \beta E[V_1(\alpha', s')|s]\right].$$

The FOC wrt H:

$$-1 + \beta z p(s) E[V_1(\alpha', s')|s] = 0.$$

So

$$(1b)\ 1 = \beta z p(s) E[V_1(\alpha', s')|s]$$

The FOC wrt m:

$$\beta = E[V_1(\alpha', s')|s] + \mu(s) - \lambda(s) = 0.$$

So

$$(2)\ \lambda(s) = \mu(s) + \beta E[V_1(\alpha', s')|s]$$

The FOC wrt n:

$$(3)\ \beta E[V_1(\alpha', s')|s] - \lambda(s) q_b(s) = 0.$$

The EC:

$$(4)\ V_1(\alpha, s) = \lambda(s)/\omega.$$

(c) We look for an equilibrim in which quantities (such as C and H) and normalized prices (such as p and q) are functions only of the current state. Equilibrium conditions are $m = 1$, $n = 0$, and $C = zH$. We conjecture that in equilibrium, no excess cash is held, so the cash-in-advance condition holds with equality: $C = 1/p(s)$. Rearrange your first-order conditions and envelope condition to answer the following.

i. *(Determination of good price)* Show that $p(s) = \frac{1}{zH(s)}$.

ii. *(Determination of labor supply)* Provide a solution for $H(s)$, with z', ω' and $H(s')$:

$$H(s) = \beta E\left[\boxed{\quad ? \quad}\middle|\, s\right].$$

iii. *(Determination of bond price)* Provide a solution for $q_b(s)$, with z, z', ω', $H(s)$ and $H(s')$:

$$q_b(s) = \beta E\left[\boxed{\quad ? \quad}\middle|\, s\right].$$

Answer: *<Step 1> Eliminate $V_1(\alpha, s)$ using*

$$\text{(4): } V_1(\alpha, s) = \lambda(s)/\omega.$$
$$\text{So: } V_1(\alpha', s') = \lambda(s')/\omega'$$

$$\text{(1a): } C^{-\theta} = p(s)\left[\mu(s) + \beta E[\lambda(s')/\omega'|s]\right].$$
$$\text{(1b): } 1 = \beta z p(s) E[\lambda(s')/\omega'|s].$$
$$\text{(2): } \lambda(s) = \mu(s) + \beta E[\lambda(s')/\omega'|s].$$
$$\text{(3): } \beta E[\lambda(s')/\omega'|s] - \lambda(s) q_b(s) = 0.$$

<Step 2> Eliminate $\mu(s)$ using (2):

$$\text{(1a): } C^{-\theta} = p(s)\lambda(s).$$
$$\text{(1b): } 1 = \beta z p(s) E[\lambda(s')/\omega'|s]. \text{ (same)}$$
$$\text{(3): } \beta E[\lambda(s')/\omega'|s] - \lambda(s) q_b(s) = 0. \text{ (same)}$$

<Step 3> Eliminate $\lambda(s)$ using

$$\text{(1a): } \lambda(s) = C^{-\theta}/p(s).$$
$$\text{So: } \lambda(s') = (C')^{-\theta}/p(s').$$

$$\text{(1b): } 1 = \beta z p(s) E\left[\left.\frac{(C')^{-\theta}}{\omega' p(s')}\right| s\right].$$
$$\text{(3): } \beta E\left[\left.\frac{(C')^{-\theta}}{\omega' p(s')}\right| s\right] - \frac{C^{-\theta}}{p(s)} q_b(s) = 0.$$

The CIA condition $C = 1/p(s)$ (and hence $C' = 1/p(s')$) imply

$$\text{(1b): } 1 = \frac{z}{C}\beta E\left[\left.\frac{(C')^{1-\theta}}{\omega'}\right| s\right].$$
$$\text{(3): } \beta E\left[\left.\frac{(C')^{1-\theta}}{\omega'}\right| s\right] - C^{1-\theta} q_b(s) = 0.$$

Change the notation - C' is replaced by $C(s')$ and C by $C(s)$. (There is a function, $C(s)$, from s to consumption.) Then,

$$\text{(1b): } 1 = \frac{z}{C(s)}\beta E\left[\left.\frac{(C(s'))^{1-\theta}}{\omega'}\right| s\right].$$
$$\text{(3): } \beta E\left[\left.\frac{(C(s'))^{1-\theta}}{\omega'}\right| s\right] - C(s)^{1-\theta} q_b(s) = 0.$$

The CIA constraint is already mentioned in the question:

$$C(s) = 1/p(s)$$

Rearranging these three equations with $C(s) = Y(s) = zH(s)$ *and* $C(s') = Y(s') = z'H(s')$,

$$H(s) = \beta E\left[\left.\frac{(z'H(s'))^{1-\theta}}{\omega'}\right| s\right] \text{ (determination of } H\text{'s)}$$

$$q_b(s) = \beta E\left[\left.\frac{1}{\omega'}\left(\frac{z'H(s')}{zH(s)}\right)^{1-\theta}\right| s\right] \text{ (determination of bond price)}$$

$$p(s) = \frac{1}{zH(s)} \text{ (determination of good price)}$$

(d) Characterize the optimal monetary policy which can attain the Pareto optimum you derived in (a). In particular,

i. Provide a mathematical condition that the (stochastic) money growth rate should satisfy to attain the Pareto optimum.

ii. What is the price of nominal bond under this optimal monetary policy?

iii. [Friedman Rule] What is the nominal interest rate under this optimal monetary policy?

Answer: *We want* $H^*(z) = z^{(1-\theta)/\theta}$. *We look for* ω' *that provides* $H^*(z) = z^{(1-\theta)/\theta}$ *as the solution for the equation,* $H(s) = \beta E\left[\left.\frac{(z'H(s'))^{1-\theta}}{\omega'}\right| s\right]$, *that we derived in (c). The easiest way to do this is to insert* $H^*(z) = z^{(1-\theta)/\theta}$ *into the equation.*

$$z^{(1-\theta)/\theta} = \beta E\left[\left.\frac{\left(z'\left(z'\right)^{(1-\theta)/\theta}\right)^{1-\theta}}{\omega'}\right| s\right].$$

$$\text{So } z^{(1-\theta)/\theta} = \beta E\left[\left.\frac{\left(z'\right)^{(1-\theta)/\theta}}{\omega'}\right| s\right]$$

So money growth rate is stochastic, satisfying the above condition. In other words, if ω' *satisfies the above condition, then we have* $H^*(z) = z^{(1-\theta)/\theta}$ *which is Pareto optimal! This is the optimal monetary policy that eliminates the distortion of cash-in-advance constraint.*

Then, with $H^*(z) = z^{(1-\theta)/\theta}$, the bond price is determined as

$$
\begin{aligned}
q_b(s) &= \beta E\left[\frac{1}{\omega'}\left(\frac{z'H(s')}{zH(s)}\right)^{1-\theta}\middle| s\right] \\
&= \beta E\left[\frac{1}{\omega'}\left(\frac{z'\,(z')^{(1-\theta)/\theta}}{zz^{(1-\theta)/\theta}}\right)^{1-\theta}\middle| s\right] \\
&= \beta E\left[\frac{(z')^{(1-\theta)/\theta}}{\omega'}\frac{1}{z^{(1-\theta)/\theta}}\middle| s\right] \\
&= \frac{1}{z^{(1-\theta)/\theta}}\beta E\left[\frac{(z')^{(1-\theta)/\theta}}{\omega'}\middle| s\right] \\
&= 1
\end{aligned}
$$

since $z^{(1-\theta)/\theta} = \beta E\left[\frac{(z')^{(1-\theta)/\theta}}{\omega'}\middle| s\right]$ in this optimal monetary policy. Bond price is one. Finally,

$$i(s) = \frac{1}{q_b(s)} - 1 = 0$$

This ($i = 0$) is known as the Friedman rule.

2. In this problem, you will consider a variation of the cash-in-advance model we discussed in class. The representative consumer has preferences:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\frac{1}{1-\sigma}\left[C_t\left(\frac{M_t}{P_tC_t}\right)^{\phi}\right]^{1-\sigma}\right],$$

for $0 < \beta < 1$, $\sigma > 0$, $\sigma \neq 1$, and $\phi > 0$, where C_t is consumption (in units of physical goods), M_t is money holding (in dollars), and P_t is the unit price of physical good (in dollars). Notice that if ϕ becomes zero, then the preferences collapse to the constant relative risk aversion (CRRA) with consumption only. The term $\frac{M_t}{P_tC_t}$ can be interpreted as the real money balance normalized by consumption.

   (a) The following is a sequence of events at each period t:

   1. The central bank injects (or withdraws) money. Denote by $\omega_t - 1$ the net growth rate of money supply (i.e., $\omega_t \equiv \overline{M}_t/\overline{M}_{t-1}$ where $\overline{M}_t$ is the money supply).

   2. The consumer allocates her asset between cash holding, M_t, and the holding of (nominal) bond, N_t, at price Q_t. To be specific, she pays Q_tN_t dollars to purchase N_t units of bonds, and at period $t+1$, N_t dollars are delivered to her.

3. Consumption is determined.

Note: Unlike the set-up in class, the cash-in-advance constraint, $P_tC_t \leq M_t$, is no longer required because cash holding is already included in the preferences. The consumer's asset at $t+1$ consists of cash and maturing bonds, plus net receipts from the sale of physical goods at the end of t, plus an injection of cash at the beginning of $t+1$. This implies that the budget constraint at $t+1$ becomes

$$M_{t+1} + Q_{t+1}N_{t+1} \leq M_t + N_t + P_t(Y_t - C_t) + (\omega_{t+1} - 1)\overline{M}_t.$$

The state vector for the economy, $s_t = (Y_t, \omega_t)$, is a first-order Markov process with a transition function $F(s'|s)$ for all s. We look at equilibria in which $P_t = \overline{M}_t p(s_t)$ and $Q_t = q(s_t)$. Define $m_t \equiv \frac{M_t}{\overline{M}_t}$ and $n_t \equiv \frac{N_t}{\overline{M}_t}$. The Bellman equation is then

$$V(\alpha, s) = \max_{C,m,n}\left[\frac{1}{1-\sigma}\left[C^{1-\phi}\left(\frac{m}{p(s)}\right)^{\phi}\right]^{1-\sigma} + \beta E\left[V(\alpha', s')|s\right]\right]$$

subject to

$$m + q(s)n \leq \alpha/\omega,$$
$$\alpha' = m + n + p(s)(Y - C) + \omega' - 1.$$

Obtain the first-order conditions and the envelope condition.

Hint: Problem: $V(x,s) = \max_y F(x,y,s) + \beta E[V(g(x,y,s),s')|s]$ s.t. $G(x,y,s) \geq 0$ and $x' = g(x,y,s)$.

- First-order Condition: $F_2(x,y,s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial y}|s\right] + \lambda(s)G_2(x,y,s) = 0$

- Envelope Condition: $V_1(x,s) = F_1(x,y,s) + \beta E\left[\frac{\partial V(g(x,y,s),s')}{\partial x}|s\right] + \lambda(s)G_1(x,y,s)$

Answer: *Eliminating* α',

$$V(\alpha, s) = \max_{C,m,n}\left[\frac{1}{1-\sigma}C^{(1-\phi)(1-\sigma)}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)} + \beta E\left[V(\underbrace{m + n + p(s)(Y-C) + \omega' - 1}_{=\alpha'}, s')|s\right]\right]$$

s.t.

$$a/\omega - m - q(s)n \geq 0. \quad [\lambda(s)]$$

The FOC wrt C:

$$\underbrace{(1-\phi)C^{(1-\phi)(1-\sigma)-1}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)}}_{\text{MU of } C} - \beta p(s)E[V_1(\alpha', s')|s] = 0.$$

Note: V_1: Derivative wrt the first element. So

$$(1)\ \underbrace{(1-\phi)C^{(1-\phi)(1-\sigma)-1}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)}}_{\text{MU of } C} = \beta \times \underbrace{p(s)E[V_1(\alpha', s')|s]}_{(*)}.$$

where (*)=increase in tomorrow's discounted expected utility when one unit of C is instead sold for dollars. The FOC wrt m:

$$(2)\ \underbrace{\phi C^{(1-\phi)(1-\sigma)}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)-1}\frac{1}{p(s)}}_{\text{MU of cash holding } m} + \beta\times \underbrace{E[V_1(\alpha', s')|s]}_{(**)} = \lambda(s).$$

where (**)=marginal increase in tomorrow's discounted expected utility when cash holding increases. The FOC wrt n:

$$(3)\ \beta E[V_1(\alpha', s')|s] - \lambda(s)q(s) = 0.$$

The EC:

$$(4)\ V_1(\alpha, s) = \lambda(s)/\omega.$$

(b) We look for an equilibrium in which quantities (such as C) and normalized prices (such as p and q) are functions only of the current state. Equilibrium conditions are $m = 1$, $n = 0$, and $C = Y$. Rearrange your first-order conditions and envelope condition to show that $p(s)$ and $q(s)$ are related as

$$p(s) = \left(\frac{1}{q(s)} - 1\right)\frac{1-\phi}{\phi}\frac{1}{Y(s)},$$

where $Y(s)$ is the output when the current shock is s. Note: If you believe the above equation is wrong, show it.

Answer: *<Step 1> Eliminate $V_1(\alpha, s)$ using*

$$\begin{aligned}(4)&:\ V_1(\alpha, s) = \lambda(s)/\omega.\\ \text{So}&:\ V_1(\alpha', s') = \lambda(s')/\omega'.\end{aligned}$$

$$(1)\ \underbrace{(1-\phi)C^{(1-\phi)(1-\sigma)-1}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)}}_{\text{MU of } C} = \beta \times \underbrace{p(s)E[\lambda(s')/\omega'|s]}_{(*)}.$$

$$(2)\ \underbrace{\phi C^{(1-\phi)(1-\sigma)}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)-1}\frac{1}{p(s)}}_{\text{MU of cash holding } m} + \beta\times \underbrace{E[\lambda(s')/\omega'|s]}_{(**)} = \lambda(s).$$

$$(3)\ \beta E[\lambda(s')/\omega'|s] - \lambda(s)q(s) = 0.$$

<Step 2> Eliminate $\lambda(s)$ *using (3):*

$$(1)\ \text{(same)}\ \underbrace{(1-\phi)C^{(1-\phi)(1-\sigma)-1}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)}}_{\text{MU of } C} = \beta \times \underbrace{p(s)E[\lambda(s')/\omega'|s]}_{(*)}.$$

$$(2)\ \underbrace{\phi C^{(1-\phi)(1-\sigma)}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)-1}\frac{1}{p(s)}}_{\text{MU of cash holding } m} + \beta \times \underbrace{E[\lambda(s')/\omega'|s]}_{(**)} = \beta \times \underbrace{\frac{1}{q(s)}E[\lambda(s')/\omega'|s]}_{(***)},$$

where (***)=increase in tomorrow's discounted expected utility when bond is purchased instead of cash holding.

<Step 3> Now eventually eliminate $\beta E[\lambda(s')/\omega'|s]$*:*

$$(2)\ \underbrace{\phi C^{(1-\phi)(1-\sigma)}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)-1}\frac{1}{p(s)}}_{\text{MU of cash holding } m} = \left(\frac{1}{q(s)}-1\right)\frac{1}{p(s)}\underbrace{(1-\phi)C^{(1-\phi)(1-\sigma)-1}\left(\frac{m}{p(s)}\right)^{\phi(1-\sigma)}}_{\text{MU of } C}.$$

Rearranging,

$$\phi C = \left(\frac{1}{q(s)}-1\right)(1-\phi)\frac{m}{p(s)}.$$

$$\text{So } p(s) = \left(\frac{1}{q(s)}-1\right)\frac{1-\phi}{\phi}\frac{m}{C}.$$

Using $m=1$ *and* $C=Y(s)$*,*

$$p(s) = \left(\frac{1}{q(s)}-1\right)\frac{1-\phi}{\phi}\frac{1}{Y(s)}.$$

3. The representative consumer has the following linear-quadratic preferences:

$$E_t\left[\sum_{\tau=0}^{\infty}\beta^{\tau}\left(-\frac{1}{2}\right)(C_{t+\tau}-C^*)^2\right]$$

for all $t = 0, 1, 2, ...$, where C_t is the units of physical goods consumed (at period t), $0 < \beta < 1$ is a time discount factor, and C^* is a constant representing a bliss point (or satiation level). Assume $C_{t+\tau} < C^*$ for all $\tau = 0, 1, ...$ The production technology is described as

$$Y_t = A_t K_t^{\alpha},$$

for all $t = 0, 1, 2, ...$, where Y_t is units of physical goods produced, A_t is stochastic productivity following a Markov process, and K_t is the physical capital stock in units of physical goods. Here, $0 < \alpha < 1$ is a constant. Notice that A_t captures all the uncertainty in this economy. At the beginning of period $t = 0, 1, 2, ...$, the consumer observes the realized value of A_t, produces the output, and makes a decision on consumption, C_t, versus investment, I_t:

$$Y_t = C_t + I_t.$$

The physical capital stock is accumulated as

$$K_{t+1} = I_t + (1 - \delta)K_t,$$

for all $t = 0, 1, 2, ...$, where $0 < \delta < 1$ is a depreciation rate. Eliminating all the constraints, we can write the Bellman equation as

$$V(K, A) = \max_{K'} \left[\left(-\frac{1}{2} \right) (AK^\alpha + (1 - \delta)K - K' - C^*)^2 + \beta E \left[V(K', A') | A \right] \right].$$

Here, as usual, K' and A' imply the physical capital stock and productivity at the next period. Assume that technical conditions to solve this problem, if any, hold. We also assume that I_t is allowed to be negative (even though you are not likely to need this).

(a) Obtain the first-order condition and the envelope condition.
Hint: Problem: $V(x, s) = \max_y F(x, y, s) + \beta E[V(g(x, y, s), s')|s]$ where $x' = g(x, y, s)$.
- First-order Condition: $F_2(x, y, s) + \beta E \left[\frac{\partial V(g(x,y,s),s')}{\partial y} | s \right] = 0$
- Envelope Condition: $V_1(x, s) = F_1(x, y, s) + \beta E \left[\frac{\partial V(g(x,y,s),s')}{\partial x} | s \right]$

Answer: *The three constraints can be rewritten as one constraint, eliminating Y_t and I_t:*

$$K_{t+1} = A_t K_t^\alpha - C_t + (1 - \delta)K_t.$$

So

$$V(K, A) = \max_{C,\, K'} \left(-\frac{1}{2} \right) (C - C^*)^2 + \beta E \left[V(K', A') | A \right]$$

subject to

$$K' = AK^\alpha - C + (1 - \delta)K.$$

We can further eliminate C :

$$V(K, A) = \max_{K'} \left(-\frac{1}{2} \right) (\underbrace{AK^\alpha + (1 - \delta)K - K'}_{=C} - C^*)^2 + \beta E \left[V(K', A') | A \right]$$

This is an unconstrained Bellman eq. The FOC is

$$(1) \quad (C - C^*) + \beta E[V_1(K', A')|A] = 0.$$

The EC is

$$V_1(K, A) = -(C - C^*)(A\alpha K^{\alpha-1} + 1 - \delta).$$

This implies

$$(2) \quad V_1(K', A') = -(C' - C^*)(A'\alpha K'^{\alpha-1} + 1 - \delta).$$

(b) Using the two conditions that you obtained in (a), provide a single equation relating C to C' (and some other variables). Of course, C and C' are the consumptions at this period and at the next period. Discuss how your equation can be interpreted as a form of the fundamental equation of asset pricing. In particular, which specific asset is priced in your equation? (What is the price of this asset in this period? What is the payoff to be delivered in the next period? How would you "name" this asset considering this economic interpretation?)

Answer: *Inserting (2) to (1),*

$$(C - C^*) = \beta E[(C' - C^*)(A'\alpha K'^{\alpha-1} + 1 - \delta)|A].$$

Or equivalently,

$$1 = E\left[\underbrace{\beta\frac{-(C' - C^*)}{-(C - C^*)}}_{(A)}(A'\alpha K'^{\alpha-1} + 1 - \delta)|A\right].$$

Since $-(C - C^*)$ is the marginal utility, (A) is the stochastic discount factor for this linear-quadratic utility. This discount factor was denoted by m'.
The fundamental eq. for asset j is $P_0^j = E_0[m_1 X_1^j]$ or $1 = E_0[m_1(1 + r_1^j)]$. So in the above equation, we are pricing an asset whose stochastic return is $A'\alpha K'^{\alpha-1} - \delta$ (marginal product of K minus depreciation). Therefore, this asset is the "investment to this economy's production (contribution to I_t)" providing the net marginal product as a return.

(c) Use your equation in (b) to answer the following.

i. Provide an explicit expression for a risk-free rate (r^f). That is, complete the expression: $\frac{1}{1+r^f} = \boxed{\quad ? \quad}$. Feel free to use C and C' (and other variables if required).

ii. Suppose that there is an asset that delivers a consumption stream. That is, if a consumer owns one unit of this asset, she is delivered C_t, C_{t+1}, C_{t+2}, ... units of physical goods at periods t, $t+1$, $t+2$, ..., respectively. Of course, C_t, C_{t+1}, C_{t+2}, ... satisfiy the conditions that you derived in (a). Go as far as you can to show that the price of this asset at period t, P_t, satisfies

$$P_t = E_t\left[\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{C_{t+\tau}^2 - C_{t+\tau}C^*}{C_t - C^*}\right].$$

Answer: *For the risk-free asset, you pay* $\frac{1}{1+r^f}$ *at* t *to receive 1 at* $t+1$. *So using our version of the fundamental eq.,*

$$\frac{1}{1+r^f} = E\left[\beta \frac{-(C' - C^*)}{-(C - C^*)}|A\right].$$

Now consider an asset that delivers a consumption stream. The fundamental equation for a one-period asset j *is* $P_t^j = E_t[m_{t+1}X_{t+1}^j]$. *For a multi-period asset* k, *we can show that it is*

$$P_t^k = E_t\left[\sum_{\tau=1}^{\infty} m_{t+\tau}X_{t+\tau}^k\right].$$

(In class, we derived a similar expression when we discuss the CAPM. To obtain this, write an optimization problem for this consumer choosing the units of asst k *to hold. Please look up your note.) Applying this,*

$$P_t = E_t\left[\sum_{\tau=1}^{\infty} m_{t+\tau}C_{t+\tau}\right],$$

where $C_{t+\tau}$ *is the consumption which already satisfies the conditions that we derived. And we know* $m_{t+\tau} = \beta^{\tau}\frac{-(C_{t+\tau}-C^*)}{-(C_t-C^*)}$ *from (b). Hence,*

$$\begin{aligned} P_t &= E_t\left[\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{-(C_{t+\tau} - C^*)}{-(C_t - C^*)} C_{t+\tau}\right] \\ &= E_t\left[\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{C_{t+\tau}^2 - C_{t+\tau}C^*}{C_t - C^*}\right] \end{aligned}$$

4. *(Permanent Income Hypothesis)* The representative consumer has the following linear-quadratic preferences:

$$E_t\left[\sum_{\tau=0}^{\infty} \beta^{\tau} u(C_{t+\tau})\right] = E_t\left[\sum_{\tau=0}^{\infty} \beta^{\tau}\left(-\frac{1}{2}\right)(C_{t+\tau} - C^*)^2\right],$$

for all $t = 0, 1, 2, ...$, where C_t is the units of physical goods consumed at period t, $0 < \beta < 1$ is a time discount factor, and C^* is a constant representing a bliss point (or satiation level). Assume $C_{t+\tau} < C^*$ for all $\tau = 0, 1, ...$ The constraint of this consumer is

$$A_{t+1} = (1+r)(A_t + Y_t - C_t),$$

where A_t is the asset holding at the beginning of period t (determined from the last period's decisions), Y_t is a stochastic endowment realized at the beginning of period t, which follows a first-order Markov process, and r is a constant interest rate. Hence, we can write the (unconstrained) Bellman equation as

$$V(A,Y) = \max_{A'} \left[u\left(\frac{A'}{1+r} - A - Y \right) + \beta E[V(A', Y')|Y] \right].$$

As usual, A' and Y' imply asset holding and endowment for the next period. Assume that technical conditions to solve this problem, if any, hold.

(a) Obtain the first-order condition and the envelope condition:

First-order condition: $\beta E[V_1(A', Y')|Y] = \boxed{\quad ? \quad}$

Envelope condition: $V_1(A, Y) = \boxed{\quad ? \quad}$

Hint: Problem: $V(x,s) = \max_y F(x,y,s) + \beta E[V(g(x,y), s')|s]$ where $x' = g(x,y)$.

- First-order Condition: $F_2(x,y,s) + \beta E\left[\frac{\partial V(g(x,y),s')}{\partial y}|s\right] = 0$

- Envelope Condition: $V_1(x,s) = F_1(x,y,s) + \beta E\left[\frac{\partial V(g(x,y),s')}{\partial x}|s\right]$

Answer: *The problem is*

$$V(A,Y) = \max_{A'} \left[\left(-\frac{1}{2}\right) \left(\frac{A'}{1+r} - A - Y - C^* \right)^2 + \beta E[V(A', Y')|Y] \right].$$

The FOC is

$$-\frac{1}{1+r} \left(\frac{A'}{1+r} - A - Y - C^* \right) + \beta E[V_1(A', Y')|Y] = 0.$$

$$\text{So (1)} \quad \frac{1}{1+r} \left(\frac{A'}{1+r} - A - Y - C^* \right) = \beta E[V_1(A', Y')|Y]$$

The EC is

$$\text{(2)} \quad V_1(A,Y) = \frac{A'}{1+r} - A - Y - C^*.$$

(b) Assume $\beta(1+r) = 1$ for convenience. Using the two conditions in (a) and $A' = (1+r)(A+Y-C)$, provide an equation relating C (this period's consumption) to C' (next period's consumption) as

$$C = E\left[\boxed{\quad ? \quad}\middle| Y\right].$$

That is, fill the blank with C' and other parameters if necessary. Interpret your results in plain English.

Answer: *Since* $A' = (1+r)(A+Y-C)$, *i.e.,* $C = Y + A - \frac{A'}{1+r}$, *(1) and (2) can be written as*

$$(1)\ \frac{1}{1+r}(-C - C^*) = \beta E[V_1(A', Y')|Y]$$

$$(2)\ V_1(A, Y) = -C - C^*$$

Here, (2) implies

$$V_1(A', Y') = -C' - C^*.$$

Plugging this to (1),

$$\frac{1}{1+r}(-C - C^*) = \beta E\left[-C' - C^* | Y\right].$$

Since $\beta(1+r) = 1$,

$$C = E\left[C' | Y\right].$$

Hence, consumption follows martingale. Today's consumption is equalized to the expected value of tomorrow's consumption.

(c) Show that consumption (C) is proportional to the present value of expected values of all future endowments plus today's asset holding ($A + Y + \frac{E[Y'|Y]}{1+r} + \frac{E[Y''|Y]}{(1+r)^2} + ...$).

Answer: *Today's consumption is equalized to the expected value of tomorrow's consumption. That is, (expected) consumption is "equalized" across the periods. Since* $C = Y + A - \frac{A'}{1+r}$ *and* $A' = \frac{A''}{1+r} - Y' + C'$,

$$\begin{aligned} C &= A + Y - \frac{-Y' + C' + \frac{A''}{1+r}}{1+r} \\ &= A + Y + \frac{Y'}{1+r} - \frac{C'}{1+r} - \frac{A''}{(1+r)^2} \\ &= A + Y + \frac{Y'}{1+r} - \frac{C'}{1+r} - \frac{-Y'' + C'' + \frac{A'''}{1+r}}{(1+r)^2} \\ &= A + Y + \frac{Y'}{1+r} + \frac{Y''}{(1+r)^2} - \frac{C'}{1+r} - \frac{C''}{(1+r)^2} - \frac{A'''}{(1+r)^3} \end{aligned}$$

Continuing in this way,

$$C + \frac{C'}{1+r} + \frac{C''}{(1+r)^2} + ... = A + Y + \frac{Y'}{1+r} + \frac{Y''}{(1+r)^2} + ...$$

Taking expectation,

$$C + \frac{E[C'|Y]}{1+r} + \frac{E[C''|Y]}{(1+r)^2} + ... = A + Y + \frac{E[Y'|Y]}{1+r} + \frac{E[Y''|Y]}{(1+r)^2} + ...$$

But from (b), we have $C = E\left[C'|\, Y\right] = E\left[C''|\, Y\right] = ...$*, so*

$$\begin{aligned} C + \frac{C}{1+r} + \frac{C}{(1+r)^2} + ... &= A + Y + \frac{E[Y'|Y]}{1+r} + \frac{E[Y''|Y]}{(1+r)^2} + ... \\ &= C\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + ...\right) \\ &= C\frac{1}{1-\frac{1}{1+r}} \\ &= C\frac{1+r}{r} \end{aligned}$$

Hence,

$$C = \frac{r}{1+r}\left[A + Y + \frac{E[Y'|Y]}{1+r} + \frac{E[Y''|Y]}{(1+r)^2} + ...\right]$$

Term Structure of Interest Rates

References

- Cochrane, J. H. (2005), Asset Pricing, Revised Edition, Princeton University Press, Princeton, NJ, Chapter 19.
- [O] Ang, A., M. Piazzesi, and M. Wei (2006), "What Does the Yield Curve Tell Us about GDP Growth?", Journal of Econometrics, 131(1-2), 359-403.

- **Data**: "Government Bonds" in EconS 320
- Usually, the "term structure of interest rates" means the "structure of term premiums of nominal annualized returns (=yields) on government bonds".
- Start from a simple set-up: No default risk, no inflation.
- $r_t^{(\tau)}$: return on government bonds, from year t to year $t+\tau$
  (Our previous notation, r_t^f, is the same as $r_t^{(1)}$ here.)
- period t period $t+1$ period $t+2$

  +————————+————————+————————+

  ↑

  We are here

  \$1 - - - - - - - - - - - > $\$1 + r_t^{(1)}$

  $\$1/(1+r_t^{(1)})$ - - - - - > \$1

  $\$1/(1+r_t^{(2)})$ - - - - - - - - - - - - - - - - - - - > \$1

  $\$1/(1+r_{t+1}^{(1)})$ - - - - - > \$1

  (not observable at period t)

- **Problem**: How are $r_t^{(1)}$ and $r_t^{(2)}$ related?
- $y_t^{(\tau)}$: **annualized** return ("**yield**") on government bonds, from t to $t+\tau$.

- By definition, $(1+y_t^{(\tau)})^\tau = 1+r_t^{(\tau)}$.
- **Problem**, re-stated: How are $y_t^{(1)}$ and $y_t^{(2)}$ related?

Contents

1. Fundamental Equation

- (1):

$$P_t^j = E_t[m_{t+1}X_{t+1}^j], \text{ where } m_{t+1} = \beta\frac{u'(Y_{t+1})}{u'(Y_t)}.$$

- $r_t^{(1)}$ satisfies

$$\text{(A)}\ \frac{1}{1+r_t^{(1)}} = E_t[m_{t+1}] = E_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}\right].$$

- What about $r_t^{(2)}$? Write the Fundamental Eq. with a two-period asset:

$$\max_a u(Y_t - a\widetilde{P}_t) + E_t\left[\beta u(Y_{t+1}) + \beta^2 u(Y_{t+2} + a\widetilde{X}_{t+2}) + \beta^3 u(Y_{t+3}) + ...\right]$$

  We have

$$\widetilde{P}_t = E_0[\widetilde{m}_{t+2}\widetilde{X}_{t+2}], \text{ where } \widetilde{m}_{t+2} = \beta^2\frac{u'(Y_{t+2})}{u'(Y_t)}.$$

- $r_t^{(2)}$ satisfies

$$\text{(B)}\ \frac{1}{1+r_t^{(2)}} = E_t[\widetilde{m}_{t+2}] = E_0\left[\beta^2\frac{u'(Y_{t+2})}{u'(Y_t)}\right]$$

- Alternatively, if we want to stick to a one-period version of Fundamental Eq., consider

$$\frac{1}{1+r_t^{(2)}} = E_t[m_{t+1}\times(?)]$$

- After one period, this two-period asset becomes exactly the same as a one-period asset newly issued at that time:

$$\frac{1}{1+r_t^{(2)}} = E_t\left[m_{t+1}\frac{1}{1+r_{t+1}^{(1)}}\right]$$

- Hence,

$$\begin{aligned}
\text{(C)}\ \frac{1}{1+r_t^{(2)}} &= E_t\left[m_{t+1}\frac{1}{1+r_{t+1}^{(1)}}\right] = E_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}\frac{1}{1+r_{t+1}^{(1)}}\right] \\
&= E_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}\right]E_t\left[\frac{1}{1+r_{t+1}^{(1)}}\right] + cov_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}, \frac{1}{1+r_{t+1}^{(1)}}\right] \\
&= \underbrace{\frac{1}{1+r_t^{(1)}}E_t\left[\frac{1}{1+r_{t+1}^{(1)}}\right]}_{\text{pure expectation}} + \underbrace{cov_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}, \frac{1}{1+r_{t+1}^{(1)}}\right]}_{\text{risk adjustment}}
\end{aligned}$$

- To study how $r_t^{(1)}$ and $r_t^{(2)}$ are related,
  - Compute them using (A) and (B). We will focus on this.
  - Alternatively, directly use (C).

2. Application: Business Cycle and Term Structure

- Consider (A) and (B).
- $\beta = 0.95$, $\sigma = 2$ for CRRA.
- There are two states:

$$\frac{Y_{t+1}}{Y_t} - 1 = \begin{cases} 0.018 + 0.036 = 0.054 \text{ (state 1: "good")} \\ 0.018 - 0.036 = -0.018 \text{ (state 2: "bad")} \end{cases}$$

- Drop the assumption of Markov transition.
- At the beginning of time t, the consumer has "some" information that predicts future GDP growth:

	Period $t+1$	Period $t+2$
Case 1	"good": 75%, "bad": 25%	"good": 75%, "bad": 25%
Case 2	"good": 75%, "bad": 25%	"good": 25%, "bad": 75%
Case 3	"good": 25%, "bad": 75%	"good": 75%, "bad": 25%
Case 4	"good": 25%, "bad": 75%	"good": 25%, "bad": 75%
... (More)		

- Case 1:

$$\text{(A)} \quad \frac{1}{1+r_t^{(1)}} = E_t[m_{t+1}] = E_t\left[\beta\frac{u'(Y_{t+1})}{u'(Y_t)}\right] = \beta E_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right]$$
$$= 0.95\left[0.75(1.054)^{-2} + 0.25(0.982)^{-2}\right]$$
$$= 0.888$$
$$\text{So } y_t^{(1)} = r_t^{(1)} = 12.7\%$$

- And

$$\text{(B)} \quad \frac{1}{1+r_t^{(2)}} = E_t[\widetilde{m}_{t+2}] = E_t\left[\beta^2\frac{u'(Y_{t+2})}{u'(Y_t)}\right] = \beta^2 E_t\left[\left(\frac{Y_{t+2}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\right]$$

(Assume 1's state and 2's state are indendent)

$$= 0.95^2[0.75 \times 0.75 \times (1.054 \times 1.054)^{-2}$$
$$+ 0.25 \times 0.25 \times (0.982 \times 0.982)^{-2}$$
$$+ 2 \times 0.25 \times 0.25 \times (0.982 \times 0.982)^{-2}]$$
$$= 0.788$$
$$\text{So } y_t^{(2)} = (1 + r_t^{(2)})^{1/2} - 1 = 12.7\%$$

- Summary:

	Yield on one-year bond ($y_t^{(1)}$)	Yield on two-year bond ($y_t^{(2)}$)	
Case 1 (good/good)	12.7%	12.7%	Flat
Case 2 (god/bad)	12.7%	8.7%	Downward Sloping
Case 3 (bad/good)	5.0%	8.7%	Upward Sloping
Case 4 (bad/bad)	5.0%	5.0%	Flat
... (More)			

- Draw the **yield curve**.
- Fact: (1) Downward-sloping yield curve tends to predict a recession after 1-2 years.
- (2) The "yield curve" is typically upward sloping. (Wrong?)

3. Application: Slope of the Yield Curve

- How can we introduce the "slope"? That is, how can we have an expression:

$$y_t^{(2)} = (\text{some function of } y_t^{(1)})$$

- Recall:

$$\text{(A)} \quad \frac{1}{1+r_t^{(1)}} = E_t[m_{t+1}] \text{ where } m_{t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}$$

$$\text{(B)} \quad \frac{1}{1+r_t^{(2)}} = E_t[\widetilde{m}_{t+2}] \text{ where } \widetilde{m}_{t+2} \equiv \beta^2 \frac{u'(C_{t+2})}{u'(C_t)} = \beta^2 \left(\frac{C_{t+2}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{-\sigma}$$

- But we can also write

$$\widetilde{m}_{t+2} = m_{t+1} m_{t+2}$$

if we define

$$m_{t+2} \equiv \beta \frac{u'(C_{t+2})}{u'(C_{t+1})}$$

to be a one-period stochastic discount factor between $t+1$ and $t+2$. Then,

$$\frac{1}{1+r_t^{(2)}} = E_t[m_{t+1} m_{t+2}].$$

- And

$$y_t^{(1)} = r_t^{(1)},$$
$$1 + y_t^{(2)} = (1 + r_t^{(2)})^{1/2}$$

- Therefore, the term structure is a relationship between $y_t^{(1)}$ and $y_t^{(2)}$ in

$$\frac{1}{1+y_t^{(1)}} = E_t[m_{t+1}]$$
$$\frac{1}{\left(1+y_t^{(2)}\right)^2} = E_t[m_{t+1} m_{t+2}]$$

- Or equivalently,

$$y_t^{(1)} = \frac{1}{E_t[m_{t+1}]} - 1,$$
$$y_t^{(2)} = \left(\frac{1}{E_t[m_{t+1} m_{t+2}]}\right)^{1/2} - 1$$

- Need:
  - How does $\{m_t\}$ evolve over time? Then we can compute $E_t[m_{t+1}]$ and $E_t[m_{t+1}m_{t+2}]$.
- **Step 1**: Consumption growth (GDP growth)
- Try a more sophisticated set-up at this time.
- Model:

$$g_{t+1} \equiv \log C_{t+1} - \log C_t, \text{ (net growth rate)}$$
$$g_{t+1} = \mu_g + \alpha_g g_t + \varepsilon^g_{t+1}, \; \varepsilon^g_{t+1}\tilde{}\text{iid } N(0, \sigma_g^2)$$

  (Estimate μ_c, α_c and σ_c^2.)
- Data for log real per-capita consumption: Sydney Ludvigson:
  http://faculty.haas.berkeley.edu/lettau/data_cay.html
- We can use Excel, MatLab, Stata, ...
-

	A	B	C	D	E
1	195201	9.12..	(g_t)	(g_{t+1})	
2	195202	9.14..	=B2-B1	=C3	=slope(D2:D230,C2:C230)
3	195203	9.15..	...	...	=intercept(D2:D230,C2:C230)
4	195204	9.15..			
5	195301	9.16..			

- (continued)
-

F	G	H	I
	$(\hat{\mu}_g + \hat{\alpha}_g g_t)$	$(\hat{\varepsilon}^g_{t+1})$	$(\hat{\sigma}_g)$
$(\hat{\alpha}_g)$	=E$3+E$2*C2	=D2-G2	=stdev(H2:H230)
$(\hat{\mu}_g)$	...	...	

- $\hat{\alpha}_g = 0.4002$, $\hat{\mu}_g = 0.0029$, $\hat{\sigma}_g = 0.0042$.
- Note: Time is quarterly now.
- Note: Now consumption growth is positively autocorrelated. (In Mehra and Prescott's (1985) calibration, it was negatively autocorrelated.)
- **Step 2**: Evolution of $\{m_t\}$ (or $\{\log m_t\}$)

- We have

$$m_{t+1} \equiv \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma}.$$

$$\text{So } \log m_{t+1} = \log \beta - \sigma \underbrace{[\log(C_{t+1}) - \log(C_t)]}_{=g_{t+1}}$$

$$\text{So } g_{t+1} = \frac{\log \beta - \log m_{t+1}}{\sigma}$$

- Insert this to the model of consumption growth above:

$$\begin{aligned}
\frac{\log \beta - \log m_{t+1}}{\sigma} &= \mu_g + \alpha_g \frac{\log \beta - \log m_t}{\sigma} + \varepsilon^g_{t+1}. \\
\text{So } \log \beta - \log m_{t+1} &= \sigma \mu_g + \alpha_g (\log \beta - \log m_t) + \sigma \varepsilon^g_{t+1}. \\
\text{So } -\log m_{t+1} &= \sigma \mu_g - \log \beta + \alpha_g \log \beta - \alpha_g \log m_t + \sigma \varepsilon^g_{t+1} \\
&= \sigma \mu_g + (\alpha_g - 1) \log \beta - \alpha_g \log m_t + \sigma \varepsilon^g_{t+1} \\
\text{So } \log m_{t+1} &= \underbrace{-[\sigma \mu_g + (\alpha_g - 1) \log \beta]}_{\equiv \gamma} + \alpha_g \log m_t - \sigma \varepsilon^g_{t+1}
\end{aligned}$$

- Then,

$$\begin{aligned}
\log m_{t+2} &= \gamma + \alpha_g \log m_{t+1} - \sigma \varepsilon^g_{t+2} \\
&= \gamma + \alpha_g (\gamma + \alpha_g \log m_t - \sigma \varepsilon^g_{t+1}) - \sigma \varepsilon^g_{t+2} \\
&= (1 + \alpha_g) \gamma + \alpha_g^2 \log m_t - \sigma (\alpha_g \varepsilon^g_{t+1} + \varepsilon^g_{t+2})
\end{aligned}$$

- **Step 3**: Solution for $y_t^{(1)}$:

$$y_t^{(1)} = \frac{1}{E_t[m_{t+1}]} - 1.$$

- Consider an approximation: $\log(1+x) \approx x$ or equivalently, $\log(x) \approx x - 1$. (You do not have to remember this formula.) Then,

$$\begin{aligned}
y_t^{(1)} &= \frac{1}{E_t[m_{t+1}]} - 1 \approx \log \left(\frac{1}{E_t[m_{t+1}]} \right) \\
&= -\log E_t[m_{t+1}] \\
&= -\log E_t[\exp(\log m_{t+1})] \\
&= -\log E_t[\exp(\gamma + \alpha_g \log m_t - \sigma \varepsilon^g_{t+1})] \\
&= -\log E_t[\exp(\gamma + \alpha_g \log m_t) \exp(-\sigma \varepsilon^g_{t+1})] \\
&= -\log \left[\exp(\gamma + \alpha_g \log m_t) E_t[\exp(-\sigma \varepsilon^g_{t+1})] \right] \\
&= -\log [\exp(\gamma + \alpha_g \log m_t)] - \log \left[E_t[\exp(-\sigma \varepsilon^g_{t+1})] \right] \\
&= -(\gamma + \alpha_g \log m_t) - \log \left[E_t[\exp(-\sigma \varepsilon^g_{t+1})] \right]
\end{aligned}$$

- But we know when x is normal, $E[\exp(x)] = \exp[\mu_x + \frac{1}{2}\sigma_x^2]$. (You do not have to remember this formula.) Hence,

$$\begin{aligned}&\varepsilon_{t+1}^g \tilde{}\text{iid } N(0,\sigma_g^2).\\ &\text{So } -\sigma\varepsilon_{t+1}^g \tilde{}\text{iid } N(0,\sigma^2\sigma_g^2).\end{aligned}$$

Hence,

$$E_t[\exp(-\sigma\varepsilon_{t+1}^g)] = \exp[0 + \frac{1}{2}\sigma^2\sigma_g^2].$$

- Hence,

$$\begin{aligned} y_t^{(1)} &= -(\gamma + \alpha_g \log m_t) - \log\left[\exp[\frac{1}{2}\sigma^2\sigma_g^2]\right] \\ &= -(\gamma + \alpha_g \log m_t) - \frac{1}{2}\sigma^2\sigma_g^2 \end{aligned}$$

- **Step 4**: Solution for $y_t^{(2)}$.
- Same trick. Use $\log(x) \approx x - 1$. Then,

$$\begin{aligned} y_t^{(2)} &= \left(\frac{1}{E_t[m_{t+1}m_{t+2}]}\right)^{1/2} - 1 \\ &\approx \log\left(\frac{1}{E_t[m_{t+1}m_{t+2}]}\right)^{1/2} \\ &= -\frac{1}{2}\log E_t[m_{t+1}m_{t+2}] \\ &= -\frac{1}{2}\log E_t[\exp(\log(m_{t+1}m_{t+2}))] \\ &= -\frac{1}{2}\log E_t[\exp(\log m_{t+1} + \log m_{t+2})] \\ &= -\frac{1}{2}\log E_t[\exp(\gamma + \alpha_g \log m_t - \sigma\varepsilon_{t+1}^g + (1+\alpha_g)\gamma + \alpha_g^2 \log m_t - \sigma(\alpha_g\varepsilon_{t+1}^g + \varepsilon_{t+2}^g))] \\ &= -\frac{1}{2}\log E_t[\exp((2+\alpha_g)\gamma + \left(\alpha_g + \alpha_g^2\right)\log m_t - \sigma((1+\alpha_g)\varepsilon_{t+1}^g + \varepsilon_{t+2}^g))] \\ &= -\frac{1}{2}\log\left\{\exp((2+\alpha_g)\gamma + \alpha_g\left(1+\alpha_g\right)\log m_t)E_t[\exp(-\sigma((1+\alpha_g)\varepsilon_{t+1}^g + \varepsilon_{t+2}^g))]\right\} \\ &= -\frac{1}{2}((2+\alpha_g)\gamma + \alpha_g\left(1+\alpha_g\right)\log m_t) - \frac{1}{2}\log E_t[\exp(-\sigma((1+\alpha_g)\varepsilon_{t+1}^g + \varepsilon_{t+2}^g))] \end{aligned}$$

- Again, we use $E[\exp(x)] = \exp[\mu_x + \frac{1}{2}\sigma_x^2]$.

$$\begin{aligned}&\varepsilon_{t+1}^g \tilde{}\text{iid } N(0,\sigma_g^2).\\ &\text{So } -\sigma((1+\alpha_g)\varepsilon_{t+1}^g + \varepsilon_{t+2}^g)\tilde{}\text{iid } N(0,\sigma^2(1+\alpha_g)^2\sigma_g^2 + \sigma^2\sigma_g^2)\end{aligned}$$

Hence,

$$E_t[\exp(-\sigma((1+\alpha_g)\varepsilon^g_{t+1}+\varepsilon^g_{t+2}))] = \exp[0+\frac{1}{2}\sigma^2\sigma_g^2(1+(1+\alpha_g)^2)].$$

- Hence,

$$\begin{aligned} y_t^{(2)} &= -\frac{1}{2}((2+\alpha_g)\gamma+\alpha_g\,(1+\alpha_g)\log m_t) - \frac{1}{2}\log\exp[\frac{1}{2}\sigma^2\sigma_g^2(1+(1+\alpha_g)^2)] \\ &= -\frac{1}{2}((2+\alpha_g)\gamma+\alpha_g\,(1+\alpha_g)\log m_t) - \frac{1}{2}[\frac{1}{2}\sigma^2\sigma_g^2(1+(1+\alpha_g)^2)] \\ &= -\frac{1}{2}\left[(2+\alpha_g)\gamma+\frac{1}{2}\sigma^2\sigma_g^2(1+(1+\alpha_g)^2)+\alpha_g\,(1+\alpha_g)\log m_t\right] \end{aligned}$$

- **Step 5**: Relating $y_t^{(2)}$ to $y_t^{(1)}$. From Step 3,

$$y_t^{(1)} = -(\gamma+\alpha_g\log m_t) - \frac{1}{2}\sigma^2\sigma_g^2.$$
$$\text{So } \gamma+\alpha_g\log m_t = -y_t^{(1)} - \frac{1}{2}\sigma^2\sigma_g^2.$$
$$\text{So } \alpha_g\log m_t = -y_t^{(1)} - \frac{1}{2}\sigma^2\sigma_g^2 - \gamma$$

Then from Step 4,

$$\begin{aligned} y_t^{(2)} &= -\frac{1}{2}\left[(2+\alpha_g)\gamma+\frac{1}{2}\sigma^2\sigma_g^2(1+(1+\alpha_g)^2)+(1+\alpha_g)\left(-y_t^{(1)}-\frac{1}{2}\sigma^2\sigma_g^2-\gamma\right)\right] \\ &= \frac{1}{2}\,(1+\alpha_g)\,y_t^{(1)} - \frac{1}{2}\left[(2+\alpha_g)\gamma+\frac{1}{2}\sigma^2\sigma_g^2(1+(1+\alpha_g)^2)+(1+\alpha_g)\left(-\frac{1}{2}\sigma^2\sigma_g^2-\gamma\right)\right] \end{aligned}$$

- Using $\hat{\alpha}_g = 0.4002$, $\hat{\mu}_g = 0.0029$, and $\hat{\sigma}_g = 0.0042$, as well as $\sigma = 2$ (CRRA) and $\beta = 0.95^{1/4} = 0.9873$ (which is quarterly), then

$$\begin{aligned} \gamma &= -\left[\sigma\mu_g+(\alpha_g-1)\log\beta\right] \\ &= -0.0135 \end{aligned}$$

- Hence, the conclusion is

$$y_t^{(2)} = 0.7000y_t^{(1)} + 0.0067.$$

- $y_t^{(2)}$ | 45-degree
- | * +

- | + (2.23%, solved from $y = 0.7000y + 0.0067$)
- | + *
- | + *
- 0.67%+ *
- | *
- +———————————— $y_t^{(1)}$
- Hence, upward sloping when $y_t^{(1)} < 2.23\%$. Downward sloping when $y_t^{(1)} > 2.23\%$. That is,
- Yield |
- | * (2-year), when $y_t^{(1)} < 2.23\%$
- |
- | * (1-year)
- |
- +————————— Maturity
- Yield |
- | * (1-year)
- |
- | * (2-year), when $y_t^{(1)} > 2.23\%$
- |
- +————————— Maturity
- **Conclusion**: We created some interesting slope.
- **My Problem**: Where does it exactly come from in plain English? Let me know if you have a good intuition.
- **Limitation**: Obviously too simple. And still facing the risk-free rate puzzle. Need to start from a model which can solve the risk-free rate puzzle.

1. The fundamental equation of asset pricing is $P_t^j = E_t[m_{t+1}X_{t+1}^j]$, where P_t^j is the price of one-period asset j at period t, m_{t+1} is the stochastic discount factor between t and $t+1$ for one-period asset, and X_{t+1}^j is the payoff of asset j at period $t+1$. Denote by $Q_t^{(n)}$ the price of n-period real bond. That is, if one pays $Q_t^{(n)}$ units of physical good at t to purchase one unit of this bond, then one unit of physical good is delivered as payoff at period $t+n$, and no payoffs are paid in all other periods.

   (a) Why is $Q_t^{(0)} = 1$ for all t?

   Answer: *The price of one unit of physical good today is of course one unit of physical good.*

   (b) Fill the blank: $Q_t^{(n)} = E_t\left[m_{t+1}Q_{t+1}^{(\boxed{\ \ ?\ \ })}\right]$, where $n = 1, 2, ...$

   Answer: $n-1$.

2. In this problem, you will construct the term structure of *nominal* interest rates. Consider a closed, endowment economy without storage. The preferences of the representative consumer are

$$E_t\left[\sum_{\tau=0}^{\infty}\beta^{\tau}\log C_{t+\tau}\right],$$

where C_t is the units of physical goods consumed (at period t), and $0 < \beta < 1$ is a time discount factor. The consumer faces two exogenous shocks: endowment shock and inflation shock. The net growth of endowment between $t-1$ and t, denoted by $g_t \equiv Y_t/Y_{t-1} - 1$, where Y_t is the units of physical goods endowed, can be either g_h (high) or g_l (low), for all t. The inflation rate between $t-1$ and t, denoted by $\pi_t \equiv P_t/P_{t-1} - 1$, where P_t is the unit price (in dollars) of physical good, can be either π_h (high) or π_l (low), for all t. Hence, a state vector of the economy at any period t is one of the following four: state #1: ($g_t = g_h$, $\pi_t = \pi_h$), state #2: ($g_t = g_h$, $\pi_t = \pi_l$), state #3: ($g_t = g_l$, $\pi_t = \pi_h$), and state #4: ($g_t = g_l$, $\pi_t = \pi_l$).

   (a) We first obtain the fundamental equation without introducing money. Consider a one-period asset, j, and a two-period asset, k, that deliver random payoffs only at the maturity. Denote their prices, in units of physical goods at period t, by Q_t^j and $\tilde{Q}_t^k$, respectively, and their payoffs, in units of physical goods at period $t+1$ and period $t+2$, by X_{t+1}^j and $\tilde{X}_{t+2}^k$. (Hence, for example, if you pay $\tilde{Q}_t^k$ units of physical goods at period t to purchase a unit of asset k, you will be delivered $\tilde{X}^k_{t+2}$ units of physical goods at period $t+2$.) Define the problems of the representative consumer and obtain the first-order conditions to complete the fundamental equations:

$$Q^j_t = E_t\left[\boxed{\quad ? \quad} \times X^j_{t+1}\right],$$
$$\tilde{Q}^k_t = E_t\left[\boxed{\quad ? \quad} \times \tilde{X}^k_{t+2}\right].$$

Answer: *The fundamental equations are*

$$Q^j_t = E_t\left[\beta\left(1+g_{t+1}\right)^{-1} X^j_{t+1}\right],$$
$$Q^k_t = E_t\left[\beta^2\left[(1+g_{t+1})(1+g_{t+2})\right]^{-1} X^k_{t+2}\right].$$

The problem:

$$\max_a E_t\left[\sum_{\tau=0}^{\infty}\beta^\tau \log C_{t+\tau}\right]$$

s.t.

$$C_t = Y_t - aQ_t,$$
$$C_{t+1} = Y_{t+1} + aX^j_{t+1},$$
$$C_{t+2} = Y_{t+2}, \ldots$$

Solving this and applying $a = 0$, we have the first equation. For the second equation, the constraints are replaced by

$$C_t = Y_t - aP_t,$$
$$C_{t+1} = Y_{t+1},$$
$$C_{t+2} = Y_{t+2} + aX^k_{t+1},$$
$$C_{t+2} = Y_{t+2}, \ldots$$

(b) We now introduce money and consider nominal bonds as special cases of assets j and k. Let $\$q^{(1)}_t$ and $\$q^{(2)}_t$ be the prices of one-period and two-period nominal bonds at period t. (Hence, for example, if you pay $\$q^{(2)}_t$ at period t to buy a unit of two-period nominal bond, you will be delivered \$1 only at period $t+2$.) Obtain explicit equations for $q^{(1)}_t$ and $q^{(2)}_t$. To do this, start from the equations given in (a).

Answer: *Paying $\$q^{(1)}_t$ at period t is the same as paying $q^{(1)}_t/P_t$ units of physical goods at period t. Receiving \$1 at period $t+1$ is the same as receiving $1/P_{t+1}$*

units of physical goods at periods $t+1$. Inserting these into the first equation of (a),

$$q_t^{(1)}/P_t = E_t\left[\beta\,(1+g_{t+1})^{-1}\,(1/P_{t+1})\right],$$

or equivalently,

$$q_t^{(1)} = E_t\left[\beta\left[(1+g_{t+1})(1+\pi_{t+1})\right]^{-1}\right].$$

Similarly we can obtain

$$q_t^{(2)} = E_t\left[\beta^2\left[(1+g_{t+1})(1+g_{t+2})\,(1+\pi_{t+1})(1+\pi_{t+2})\right]^{-1}\right].$$

(c) Assume $\beta = 0.95$, $g_h = 4\%$, $g_l = 0\%$ (so that average annual net growth of endowment becomes 2%), $\pi_h = 5\%$, and $\pi_l = 1\%$ (so that average annual net inflation rate becomes 3%). All four states at each period occur with equal probabilities. Period-$(t+1)$ state and period-$(t+2)$ state are independent. Draw the yield curve implied by this model.

Answer: *We have to evaluate $q_t^{(1)}$ and $q_t^{(2)}$ given these numbers:*

$$\begin{aligned} q_t^{(1)} &= \beta\times(1/4,1/4,1/4,1/4)\left(\begin{pmatrix}(1+g_h)^{-1}\\(1+g_l)^{-1}\end{pmatrix}\otimes\begin{pmatrix}(1+\pi_h)^{-1}\\(1+\pi_l)^{-1}\end{pmatrix}\right)\\ &= 0.9049 \end{aligned}$$

where $\otimes$ is Kronecker product. Hence,

$$y_t^{(1)} = \frac{1}{q_t^{(1)}} - 1 = 0.1051$$

Also,

$$\begin{aligned} q_t^{(2)} &= \beta^2\times\frac{1}{16}\times\mathbf{1}_{1\times 16}\\ &\times\left[\left(\begin{pmatrix}(1+g_h)^{-1}\\(1+g_l)^{-1}\end{pmatrix}\otimes\begin{pmatrix}(1+\pi_h)^{-1}\\(1+\pi_l)^{-1}\end{pmatrix}\right)\otimes\left(\begin{pmatrix}(1+g_h)^{-1}\\(1+g_l)^{-1}\end{pmatrix}\otimes\begin{pmatrix}(1+\pi_h)^{-1}\\(1+\pi_l)^{-1}\end{pmatrix}\right)\right]\\ &= 0.8189 \end{aligned}$$

Hence, from $(1+y_t^{(2)})^2 = 1/q_t^{(2)}$,

$$y_t^{(2)} = \left(\frac{1}{q_t^{(2)}}\right)^{1/2} - 1 = 0.1051$$

Hence, the yield curve is flat at 10.5%.

If you are a MatLab user, this is the code that I use:

```
beta=0.95
Gh=1.04
Gl=1
Ph=1.05
Pl=1.01
four_states=kron([Gh^(-1);Gl^(-1)],[Ph^(-1);Pl^(-1)])
q1t=beta*(1/4)*ones(1,4)*four_states
y1t=1/q1t-1
q2t=beta^2*(1/16)*ones(1,16)*kron(four_states,four_states)
y2t=(1/q2t)^0.5-1
```

3. Consider the term structure that we considered in class:

$$\begin{aligned} y_t^{(1)} &= \frac{1}{E_t[m_{t+1}]} - 1, \\ y_t^{(2)} &= \left(\frac{1}{E_t[m_{t+1}m_{t+2}]} \right)^{1/2} - 1, \\ m_{t+1} &\equiv \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma}, \\ m_{t+2} &\equiv \beta \left(\frac{C_{t+2}}{C_{t+1}} \right)^{-\sigma}, \\ g_{t+1} &= \mu_g + \alpha_g g_t + \varepsilon_{t+1}^g, \ \varepsilon_{t+1}^g \text{~iid } N(0, \sigma_g^2), \\ g_{t+1} &\equiv \log C_{t+1} - \log C_t. \end{aligned}$$

Use the numbers that we assumed in class for calibration. Suppose that we are at 2011Q1 (first quarter of 2011) and $g_{2011Q1} \equiv \log C_{2011Q1} - \log C_{2010Q4} = 0.5\%$. Obtain the levels of $y_{2011Q1}^{(1)}$ and $y_{2011Q1}^{(2)}$ predicted by this model. Discuss your result. Note: Feel free to quote the equations that we derived in class.

Answer: *Let* $t = 2011Q1$*. Then,*

$$m_t \equiv \beta \left(\frac{C_t}{C_{t-1}} \right)^{-\sigma},$$

$$\begin{aligned} \text{So } \log m_t &= \log \beta - \sigma(\log C_t - \log C_{t-1}) \\ &= \log(0.9873) - 2(0.005) \\ &= -0.0228 \end{aligned}$$

Hence, from the equations that we derived,

$$\begin{aligned} y_t^{(1)} &= -(\gamma + \alpha_g \log m_t) - \frac{1}{2}\sigma^2\sigma_g^2 \\ &= -(-0.0135 + 0.4002(-0.0228)) - \frac{1}{2}(2)^2(0.0042)^2 \\ &= 0.0259 \end{aligned}$$

and

$$\begin{aligned} y_t^{(2)} &= 0.7000 y_t^{(1)} + 0.0067 \\ &= 0.7000 \times 0.0259 + 0.0067 \\ &= 0.0248 \end{aligned}$$

So it is a slightly downward-sloping yield curve. This is a quantitative failure – the consumption growth of 0.5% per quarter corresponds to 2% per year, which is average. So this result means that on average, the yield curve is downward sloping. This is perhaps because we are using a simple model that faces the risk-free rate puzzle.

4. Consider a closed, endowment economy with money and single, perishable physical goods. The fundamental equation of asset pricing for a one-period asset j is $P_t^j = E_t[m_{t+1}X_{t+1}^j]$, where P_t^j is the price of asset j in units of physical goods at period t, m_{t+1} is the stochastic discount factor between t and $t+1$ for one-period asset, and X_{t+1}^j is the payoff in units of physical goods at period $t+1$. For example, for log preferences, we have $m_{t+1} = \beta\,(C_{t+1}/C_t)^{-1}$, where $0 < \beta < 1$ is a time discount factor and C_t is the units of physical goods consumed at period t.

   Denote by p_t the price level at period t. That is, one unit of physical good is worth p_t dollars at period t.

   Consider the following asset prices:

   - $Q_t^{(n)}$: Period-t price of a *real* bond with a duration of n periods. That is, if one pays $Q_t^{(n)}$ *units of physical goods* at period t to purchase this asset, then this asset will deliver one *unit of physical good* at period $t+n$. (There are no other deliveries.)
   - $q_t^{(n)}$: Period-t price of a *nominal* bond with a duration of n periods. That is, if one pays $q_t^{(n)}$ *dollars* at period t to purchase this asset, then this asset will deliver one *dollar* at period $t+n$. (There are no other deliveries.)

   Make reasonable assumptions if required.

(a) Fill the blank (A) in

$$Q_t^{(n)} = E_t\left[m_{t+1}Q_{t+1}^{(\boxed{\quad (A) \quad})}\right],$$

for $n = 1, 2, \ldots$ Explain.

Answer: $\boxed{\quad (A) \quad} = n - 1$. *After one period, the duration decreases by one period.*

(b) Provide a counterpart expression to (a) for $q_t^{(n)}$. That is, fill the blank (B) in

$$q_t^{(n)} = E_t\left[m_{t+1} \times \boxed{\quad (B) \quad}\right],$$

where the blank should be a term consisting of $q_{t+1}^{(\boxed{\quad (A) \quad})}$ and other variables if necessary.

Answer: *Paying $q_t^{(n)}$ dollars at period t is the same as paying $q_t^{(n)}/P_t$ units of physical goods at period t. At period $t+1$, this asset has the value of $q_{t+1}^{(n-1)}$ dollars. This is $q_{t+1}^{(n-1)}/P_{t+1}$ units of physical goods. Inserting these into the fundamental equation, $P_t^j = E_t[m_{t+1}X_{t+1}^j]$, we have*

$$q_t^{(n)}/P_t = E_t\left[m_{t+1}q_{t+1}^{(n-1)}/P_{t+1}\right],$$

and hence,

$$q_t^{(n)} = E_t\left[m_{t+1}q_{t+1}^{(n-1)}P_t/P_{t+1}\right].$$

So $\boxed{\quad (B) \quad} = q_{t+1}^{(n-1)}P_t/P_{t+1}$.

(c) Study how $Q_t^{(1)}$ and $q_t^{(1)}$ are related. (The superscript is "(1)", not "(n)".) That is, use your results for subquestions (a) and (b) to fill the blanks, (C) and (D), in

$$q_t^{(1)} = Q_t^{(1)} \times \boxed{\quad (C) \quad} + \boxed{\quad (D) \quad}.$$

Answer based on your $\boxed{\quad (C) \quad}$ and $\boxed{\quad (D) \quad}$: If $Q_t^{(1)}$ increases, then does $q_t^{(1)}$ tend to increase? Also, is $q_t^{(1)}$ affected by the anticipations about future inflation, such as $E_t[P_t/P_{t+1}]$ or $\sigma_t[P_t/P_{t+1}]$? Explain.

Answer: *From the above two parts, $Q_t^{(1)} = E_t[m_{t+1}]$ and $q_t^{(1)} = E_t[m_{t+1}P_t/P_{t+1}]$. Hence,*

$$\begin{aligned} q_t^{(1)} &= E_t[m_{t+1}]E_t[P_t/P_{t+1}] + cov_t[m_{t+1}, P_t/P_{t+1}] \\ &= Q_t^{(1)}E_t[P_t/P_{t+1}] + \sigma_t[m_{t+1}]\sigma_t[P_t/P_{t+1}]\rho_t[m_{t+1}, P_t/P_{t+1}] \end{aligned}$$

This implies that $q_t^{(1)}$ and $Q_t^{(1)}$ are positively related. (It is also OK to argue against it if properly supported.) $q_t^{(1)}$ is indeed affected by $E_t[P_t/P_{t+1}]$, $\sigma_t[m_{t+1}]$, $\sigma_t[P_t/P_{t+1}]$ and $\rho_t[m_{t+1}, P_t/P_{t+1}]$.

5. Reference:

   Ang, A., M. Piazzesi, and M. Wei (2006), "What Does the Yield Curve Tell Us about GDP Growth?", Journal of Econometrics, 131(1-2), 359-403.

   The first paragraph of this paper is as follows.

   The behavior of the yield curve changes across the business cycle. In recessions, premia on long-term bonds tend to be high and yields on short bonds tend to be low. Recessions, therefore, have upward sloping yield curves. Premia on long bonds are countercyclical because investors do not like to take on risk in bad times. The lower demand for long bonds during recessions lowers their price and increases their yield. In contrast, yields on short bonds are procyclical because of monetary policy. The Federal Reserve lowers short yields in recessions in an effort to stimulate economic activity. For example, for every 2 percentage point decline in GDP growth, the Fed should lower the nominal yield by 1 percentage point according to the Taylor (1993) rule.

   Consider the following:

   (a) Annual yield on 3-month government bonds
   (b) Annual yield on 10-year government bonds
   (c) Term premium ((b) minus (a))

   Visit http://www.federalreserve.gov/releases/h15/data.htm, which is the database maintained by the Federal Reserve. For (a), find "Treasury bills (secondary market)" and download 3-month monthly data. For (b), find "Treasury constant maturities" and download nominal 10-year monthly data. Do not worry about the inflation – We want nominal data. Report the sample means of the above three items, during the following periods, based on NBER business cycle data. Discuss your results.

   (a) From: One year before the peak, To: Peak
   (b) From: Peak, To: One year after the peak
   (c) From: One year before trough, To: Trough
   (d) From: Trough, To: One year after Trough

Cross-Section of Asset Returns

References

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- [O] Kim, H., C. Nam, S. H. Johnson, and S. Choi (2011), "Propensity to Pay Dividends and Stock Returns," unpublished manuscript.
- [O] Lucas, R. E., Jr. (1978), "Asset Prices in an Exchange Economy," Econometrica, 46(6), 1429-1445.
- [O] Cochrane, J. H. (1999), "New Facts in Finance," Economics Perspectives, 23(3), 36-58.
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1. Orchard Model

- So far: The representative consumer receives Y_t units of "fruits" provided by a "tree," which is often called the **Lucas tree**, following Lucas (1976).
- Now we will study a model with multiple Lucas trees, or **Lucas orchard**.
- An endowment economy consists of two assets, A and B.
- Asset A is risk-free, providing constant one unit of physical good as dividend for all t.
- Asset B's dividend is stochastic:

$$D_t^B = \begin{cases} 1+e \text{ with probability } \frac{1}{2} \text{ (state 1)} \\ 1-e \text{ with probability } \frac{1}{2} \text{ (state 2)} \end{cases}$$

  where $0 < e < 1$, i.i.d. over time.
- $C_t = 1 + D_t^B$ for all t.

- The consumption share of asset A's dividend: $s_t \equiv 1/C_t$.
- Hence, s_t is either $1/(2+e)$ in state 1 or $1/(2-e)$ in state 2.
- If a risk-averse investor were to choose his own value of s_t, he would choose $s_t = 1$ to minimize the consumption volatility.
- However, this story becomes entirely different if he is a **representative** investor in this endowment economy. That is, the markets trading asset A and asset B need to be cleared, whichever value of s_t (either $1/(2+e)$ or $1/(2-e)$) is given as endowment. In an equilibrium, the prices of the two assets are determined so that the representative investor have no motives to deviate from the given value of s_t to $s_t = 1$.
- Consumption growth is

$$\frac{C_{t+1}}{C_t} = \begin{cases} 1 \text{ with probability } \frac{1}{2} \\ \frac{2-e}{2+e} < 1 \text{ with probability } \frac{1}{2} \end{cases}$$

  if state 1 in period t, and

$$\frac{C_{t+1}}{C_t} = \begin{cases} \frac{2+e}{2-e} > 1 \text{ with probability } \frac{1}{2} \\ 1 \text{ with probability } \frac{1}{2} \end{cases}$$

  if state 2 in period t, for all t.
- To see this clearly, assume the log utility
- The Euler equation suggests that asset A's price at period t, denoted by P_t^A, satisfies

$$P_t^A = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-1} (1 + P_{t+1}^A) \right].$$

  Then,

$$P_1^A = \beta \left[\frac{1}{2}(1 + P_1^A) + \frac{1}{2} \left(\frac{2-e}{2+e} \right)^{-1} (1 + P_2^A) \right],$$

$$P_2^A = \beta \left[\frac{1}{2} \left(\frac{2+e}{2-e} \right)^{-1} (1 + P_1^A) + \frac{1}{2}(1 + P_2^A) \right].$$

  Solving this equation system, one has $P_1^A = 2\beta/(1-\beta)(2-e)$ and $P_2^A = 2\beta/(1-\beta)(2+e)$. This implies that the price of asset A is higher in state 1 than in state 2.

- The result that the asset price differs between the states arises from precautionary motives of the representative investor.

- For **precautionary motives**, the investor in state 1 expects that future consumption growth will be lower, and hence, wishes to save.

- In state 1, the next period's endowment is expected to be lower than this period's. Hence, the investor is willing to purchase more units of assets. Since the demand is higher, equilibrium asset prices will be higher.

- Road map from here (not discussed in class):

  1. Obtain r^A_{t+1}. (Four possible values: "$1 \to 1$", "$1 \to 2$", "$2 \to 1$", "$2 \to 2$".)
  2. Similarly obtain P^B_t and r^B_{t+1}.
  3. Study how they move together:
     (a) $corr_t(r^A_{t+1}, r^B_{t+1})$ (XS correlation)
     (b) $corr_t(r^A_{t+1}, r^A_{t+2})$ (autocorrelation)
     (c) $corr_t(r^A_{t+1}, r^B_{t+2})$ (cross-serial correlation)
  4. Cross-sectional properties explained? Intuition?
     (a) Value premium, Size premium, ... (Kim and Nam (2011))
     (b) Betas
     (c) Predictive power of P/D ratios in XS
     (d) Momentum effect (Cochrane, Longstaff and Santa-Clara (2008))
     (e) Stock returns vs. Bond returns (Choi and Chung (2011))
     (f) Cross-country differences in international stock indices
     (g) ...
  5. Try a better model. (Start with a model that solves the equity-premium puzzle, etc., such as rare disasters, habits, Epstein-Zin, ...)

EXERCISES

1. Past short-run winners tend to dominate in the stock market. This is called the *momentum effect*.

   Visit http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html which is Kenneth French's data library. Download "10 Portfolios Formed on Momentum." This dataset provides monthly returns on ten prior-return portfolios. That is, portfolio #1 ("Low") consists of the bottom 10% stocks based on past 1-year returns. Portfolio #10 ("High") consists of the top 10%. Use "Average Value Weighted Returns – Monthly" in the dataset. Use the data from January 1946.

   (a) Transform nominal returns to real returns using CPI inflation provided by the Bureau of Labor Statistics. (Obtain a monthly inflation rate for January 1946 by comparing December-1945 and January-1946 price levels. Subtract it from January-1946 nominal returns.) Plot the sample averages of monthly real returns and the sample standard deviations of monthly real returns, for ten prior-return portfolios. Discuss the result. Is the result consistent with your economic intuition? Why or why not?

   (b) The capital asset pricing model (CAPM) can be written as

   $$E\left[r_t^j - r_t^f\right] = \beta_j E\left[r_t^m - r_t^f\right],$$

   for all j, where r_t^j is stock or portfolio j's return at period t, r_t^f is a risk-free rate, β_j is j's "beta" (regression slope), and r_t^m is the return on the aggregate market. Refer to my discussions on the CAPM at "Cross-Section of Stocks" in my ECONS 320, available at http://user.chol.com/~estudiar/English/320_09_7.pdf. Notice that j can be any stock or portfolio, such as each prior-return portfolio, government bonds, the aggregate stock market, Microsoft, etc.

   To see whether the cross-sectional returns of the prior-return portfolios can be explained by the CAPM, run the following time-series OLS regressions for each portfolio j:

   $$r_t^j - r_t^f = \alpha_j + \beta_j\left(r_t^m - r_t^f\right) + \varepsilon_t^j,$$

   where ε_t^j is an error term. For $r_t^m - r_t^f$, use "Mkt-RF" in "Fama/French Factors" in Kenneth French's website. For $r_t^j - r_t^f$, subtract "RF" (which is nominal) in "Fama/French Factors" from your nominal return on each portfolio.

i. Plot $\hat{\beta}_j$ (estimated value of β_j) and $E_T\left[r_t^j - r_t^f\right]$ (sample mean of $r_t^j - r_t^f$, where the subscript T means the sample), for ten prior-return portfolios. If the CAPM is correct, how should they fit in this plot? (Again, refer to my CAPM discussions.) Does your result match with the CAPM's implication?

ii. If the CAPM is correct, then α_j is zero for all j. Compare the expected excess returns on prior-return portfolios predicted by the CAPM (which is $E_T\left[\hat{\beta}_j\left(r_t^m - r_t^f\right)\right]$) and the same expected excess returns from the data (which is $E_T\left[r_t^j - r_t^f\right]$), for ten prior-return portfolios. Does your result match with the CAPM's implication?

iii. Suppose that you invest in some of prior-return portfolios. If the CAPM is roughly a correct model, how will you invest based on your answers to i and ii?

(c) Repeat ii and iii of (b) with the Fama-French three-factor model which is empirically more successful than the CAPM:

$$E\left[r_t^j - r_t^f\right] = \beta_j E\left[r_t^m - r_t^f\right] + \gamma_j E\left[SMB_t\right] + \delta_j E\left[HML_t\right] + \varepsilon_t^j,$$

where SMB_t and HML_t can be found in "Fama/French Factors".

2. There are two trees, A and B, in a closed, deterministic endowment economy. At period t, trees A and B provide Y_t^A and Y_t^B units of identical fruits, respectively, as endowments. The preferences of the representative consumer at period 0 are

$$\sum_{t=0}^{\infty} \beta^t \log C_t,$$

where $0 < \beta < 1$ is a discount factor, and C_t is the units of physical goods (or fruits) consumed at period t. There is no storage. Hence, the resource constraints are

$$C_0 = Y_0^A + Y_0^B,$$
$$C_1 = Y_1^A + Y_1^B,$$
$$\ldots$$

Tree A provides constantly growing endowments:

$$\frac{Y_{t+1}^A}{Y_t^A} = g^A,$$

for all $t = 0, 1, 2, ...$, where g^A is constant. For example, if a net growth rate is 2%, then $g^A = 1.02$. Tree B's endowment (fruit) growth is

$$\frac{Y_{t+1}^B}{Y_t^B} = \begin{cases} g_H^B \text{ when } t = 0, 2, 4, ... \\ g_L^B \text{ when } t = 1, 3, 5, ... \end{cases}$$

where g_H^B and g_L^B are constant and $g_H^B > g_L^B$. Define $s_t \equiv Y_t^A / C_t^A$.

(a) Consider the following asset: If you purchase one unit of this asset at period t, it delivers one unit of physical good at period $t+1$. Denote the unit price of this asset (in units of physical goods) at t by P_t^f. Provide a (closed-form) solution for P_t^f. Your solution should contain s_t, g^A, g_H^B and g_L^B only. Hint: One can easily show

$$\frac{C_{t+1}}{C_t} = \begin{cases} s_t g^A + (1 - s_t) g_H^B \text{ when } t = 0, 2, 4, ... \\ s_t g^A + (1 - s_t) g_L^B \text{ when } t = 1, 3, 5, ... \end{cases}$$

Answer: *The consumption growth is*

$$\begin{aligned} \frac{C_{t+1}}{C_t} &= \frac{Y_{t+1}^A + Y_{t+1}^B}{Y_t^A + Y_t^B} = \frac{Y_{t+1}^A}{Y_t^A + Y_t^B} + \frac{Y_{t+1}^B}{Y_t^A + Y_t^B} \\ &= \frac{Y_t^A}{Y_t^A + Y_t^B} \frac{Y_{t+1}^A}{Y_t^A} + \frac{Y_t^B}{Y_t^A + Y_t^B} \frac{Y_{t+1}^B}{Y_t^B} \\ &= s_t g^A + (1 - s_t) \frac{Y_{t+1}^B}{Y_t^B} \\ &= \begin{cases} s_t g^A + (1 - s_t) g_H^B \text{ when } t = 0, 2, 4, ... \\ s_t g^A + (1 - s_t) g_L^B \text{ when } t = 1, 3, 5, ... \end{cases} \end{aligned}$$

The fundamental eqaution of asset pricing for the log utility is $P_t^j = E_t \left[\beta (C_{t+1}/C_t)^{-1} X_{t+1}^j \right]$. *Our set-up does not have uncertainty. Hence,*

$$\begin{aligned} P_t^f &= \beta (C_{t+1}/C_t)^{-1} \\ &= \begin{cases} \frac{\beta}{s_t g^A + (1 - s_t) g_H^B} \text{ when } t = 0, 2, 4, ... \\ \frac{\beta}{s_t g^A + (1 - s_t) g_L^B} \text{ when } t = 1, 3, 5, ... \end{cases} \end{aligned}$$

(b) Is P_t^f higher or lower when $t = 0, 2, 4, ...$ than when $t = 1, 3, 5, ...$? Due to which "motives" do we have this result? Explain in plain English.

Answer: *Since* $g_H^B > g_L^B$, *we have* $\frac{\beta}{s_t g^A + (1 - s_t) g_H^B} < \frac{\beta}{s_t g^A + (1 - s_t) g_L^B}$. *The price of this asset is lower when* $t = 0, 2, 4, ...$ *This is a result of precautionary motives. When tree B's dividend growth is to be higher in the next period, the consumer has less incentives to save (since she already expects more endowments than other times), so the demand for this asset is lower. Hence, this asset's price is lower.*

(c) Obtain the price of tree B at period t, denoted by P_t^B. That is, if you purchase one unit of this asset at period t, it delivers Y_{t+1}^B units of physical goods at period $t+1$, while you still have one unit of this asset, which is valued P_{t+1}^B units of physical goods. Show that

$$\frac{P_t^B}{C_t} = \beta \left(1 - s_{t+1} + \frac{P_{t+1}^B}{C_{t+1}}\right).$$

Answer: *The fundamental eqaution of asset pricing for the log utility is* $P_t^j = E_t\left[\beta(C_{t+1}/C_t)^{-1}X_{t+1}^j\right]$. *Our set-up does not have uncertainty. Hence, for tree B, we have* $P_t^B = \beta(C_{t+1}/C_t)^{-1}(Y_{t+1}^B + P_{t+1}^B)$. *Hence,*

$$\begin{aligned}\frac{P_t^B}{C_t} &= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\left(\frac{Y_{t+1}^B}{C_t} + \frac{P_{t+1}^B}{C_t}\right)\\ &= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{C_{t+1}}{C_t}\left(\frac{Y_{t+1}^B}{C_{t+1}} + \frac{P_{t+1}^B}{C_{t+1}}\right)\\ &= \beta\left(1 - s_{t+1} + \frac{P_{t+1}^B}{C_{t+1}}\right)\end{aligned}$$

3. Interested students would look at Cochrane, J. H., F. A. Longstaff, and P. Santa-Clara (2008), "Two Trees," Review of Financial Studies, 21(1), 347-385.

   There are two Lucas trees, A and B. At period t, trees A and B provide Y_t^A and Y_t^B units of identical fruits, respectively. We match $Y_t^A + Y_t^B$ with the GDP. The preferences of the representative consumer at period 0 follows the log utility:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t \log C_t\right],$$

   where $0 < \beta < 1$ is a discount factor, C_t is the units of physical goods (or fruits) consumed at period t. There is no storage. This is a closed economy. Hence, the resource constraints are

$$\begin{aligned}C_0 &= Y_0^A + Y_0^B,\\ C_1 &= Y_1^A + Y_1^B,\\ &\cdots\end{aligned}$$

   Tree A provides constantly growing endowments (or fruits):

$$\frac{Y_{t+1}^A}{Y_t^A} = 1.02,$$

for all $t = 0, 1, 2, ...$ Tree B's endowment (fruit) growth is independent and identically distributed (i.i.d.) over time:

$$\frac{Y_{t+1}^B}{Y_t^B} = \begin{cases} 1.05 \text{ with probability } 50\% \\ 0.99 \text{ with probability } 50\% \end{cases},$$

for all $t = 0, 1, 2, ...$

(a) Consider a one-period risk-free asset. Show that the (net) rate of return on this asset, or risk-free rate, between periods 0 and 1, denoted by r_1^f, satisfies

$$\frac{1}{1+r_1^f} = E_0 \left[\beta \left(\frac{Y_1^A + Y_1^B}{Y_0^A + Y_0^B} \right)^{-1} \right]. \tag{1}$$

Answer: *We solve*

$$\max_a E_0 \left[\sum_{t=0}^{\infty} \beta^t \log C_t \right]$$

s.t.

$$C_0 = Y_0^A + Y_0^B - a \left(\frac{1}{1+r_1^f} \right),$$
$$C_1 = Y_1^A + Y_1^B + a,$$
$$C_2 = Y_2^A + Y_2^B,$$
$$\cdots$$

Of course, you can write it with P_0^f and X_1^f and later define r_1^f. The unconstrained problem is

$$\max_a \log \left[Y_0^A + Y_0^B - a \left(\frac{1}{1+r_1^f} \right) \right] + \beta E_0 \left[\log(Y_1^A + Y_1^B + a) \right] + ...$$

The FOC is

$$-\frac{1}{C_0} \left(\frac{1}{1+r_1^f} \right) + E_0 \left[\beta \frac{1}{C_1} \right] = 0.$$

$$\text{So } \frac{1}{1+r_1^f} = E_0 \left[\beta \left(\frac{C_1}{C_0} \right)^{-1} \right].$$

There is no storage and closed economy. So in equil, the above FOC holds, and $a = 0$. So $C_0 = Y_0^A + Y_0^B$ and $C_1 = Y_1^A + Y_1^B$.

(b) Define s_t to be the share of asset A's endowment out of the aggregate endowment:

$$s_t \equiv \frac{Y_t^A}{Y_t^A + Y_t^B}.$$

Assume $0 < s_t < 1$ for all t. Show that

$$\frac{Y_{t+1}^A + Y_{t+1}^B}{Y_t^A + Y_t^B} = \begin{cases} 1.05 - 0.03s_t \text{ with probability } 50\% \\ 0.99 + 0.03s_t \text{ with probability } 50\% \end{cases} \tag{2}$$

(Hint: All you need is $s_t = \frac{Y_t^A}{Y_t^A + Y_t^B}$.)

Answer:

$$\begin{aligned} \frac{Y_{t+1}^A + Y_{t+1}^B}{Y_t^A + Y_t^B} &= \frac{Y_{t+1}^A}{Y_t^A + Y_t^B} + \frac{Y_{t+1}^B}{Y_t^A + Y_t^B} \\ &= \frac{Y_{t+1}^A}{\frac{Y_t^A}{s_t}} + \frac{Y_{t+1}^B}{\frac{Y_t^B}{1-s_t}} \\ &= s_t \frac{Y_{t+1}^A}{Y_t^A} + (1-s_t)\frac{Y_{t+1}^B}{Y_t^B} \\ &= \begin{cases} 1.02s_t + (1-s_t)1.05 \text{ with prob } 50\% \\ 1.02s_t + (1-s_t)0.99 \text{ with prob } 50\% \end{cases} \\ &= \begin{cases} 1.05 - 0.03s_t \text{ with prob } 50\% \\ 0.99 + 0.03s_t \text{ with prob } 50\% \end{cases} \end{aligned}$$

(c) Using equation (2), eliminate $\frac{Y_{t+1}^A + Y_{t+1}^B}{Y_t^A + Y_t^B}$ in equation (1). That is, provide a new formula for $\frac{1}{1+r_1^f}$ as a function of s_0, without expectation (E_0). If s_0 becomes higher, does r_1^f also become higher? (Hint: Take the first derivative. Recall that we assumed $0 < s_0 < 1$.)

Answer:

$$\begin{aligned} \frac{1}{1+r_1^f} &= E_0\left[\beta\left(\frac{Y_1^A + Y_1^B}{Y_0^A + Y_0^B}\right)^{-1}\right] \\ &= \frac{\beta}{2}\frac{1}{1.05 - 0.03s_0} + \frac{\beta}{2}\frac{1}{0.99 + 0.03s_0} \end{aligned}$$

Differentiate the RHS wrt s_0 gives

$$\frac{\beta}{2}\frac{0.03}{(1.05 - 0.03s_0)^2} - \frac{\beta}{2}\frac{0.03}{(0.99 + 0.03s_0)^2}$$

If this is positive, then $\frac{1}{1+r^f}$ *increases in* s_0*. If negative, then* $\frac{1}{1+r^f}$ *decreases. When is it positive?*

$$\begin{aligned} \frac{\beta}{2}\frac{0.03}{(1.05-0.03s_0)^2} - \frac{\beta}{2}\frac{0.03}{(0.99+0.03s_0)^2} &> 0. \\ \text{So } \frac{0.03}{(1.05-0.03s_0)^2} &> \frac{0.03}{(0.99+0.03s_0)^2} \\ \text{So } (0.99+0.03s_0)^2 &> (1.05-0.03s_0)^2 \\ \text{So } 0.99+0.03s_0 &> 1.05-0.03s_0 \end{aligned}$$

since both $1.05 - 0.03s_0$ *and* $0.99 + 0.03s_0$ *are positive. This is equivalent to* $s_0 > 1$*. By assumption, this does not hold. So* $\frac{1}{1+r_1^f}$ *is decreasing. So* r_1^f *increases.*

(d) Provide an economic interpretation (in plain English) to understand your answer to (c). That is, why would the risk-free rate be affected by the dividend share of tree A, in the way you described in (c)?

Answer: *As* s_t *declines, the economy becomes more unstable since the share of risky asset, tree B, increases. The investor faces more uncertainty, and hence, he will increasingly prefer the risk-free asset. This makes the risk-free asset more expensive. Risk-free rate declines. (As* s_t *rises, on the other hand, the risk-free rate rises.)*

(e) Now we revisit (a) and consider the price of tree A. Denote the price of tree A at period t by P_t^A. Show that

$$\frac{P_0^A}{Y_0^A} = 1.02E_0\left[\beta\left(\frac{Y_1^A+Y_1^B}{Y_0^A+Y_0^B}\right)^{-1}\left(1+\frac{P_1^A}{Y_1^A}\right)\right].$$

Answer: *We go back to (a). The fundamental equation for a one-period asset* j *is*

$$P_0^j = E_0\left[\beta\left(\frac{Y_1^A+Y_1^B}{Y_0^A+Y_0^B}\right)^{-1} X_1^j\right].$$

In case of tree A, the payoff at period 1 is $Y_1^A + P_1^A$*. So*

$$P_0^A = E_0\left[\beta\left(\frac{Y_1^A+Y_1^B}{Y_0^A+Y_0^B}\right)^{-1} (Y_1^A+P_1^A)\right].$$

Dividing both sides by Y_0^A,

$$\begin{aligned}\frac{P_0^A}{Y_0^A} &= E_0\left[\beta\left(\frac{Y_1^A+Y_1^B}{Y_0^A+Y_0^B}\right)^{-1}\left(\frac{Y_1^A}{Y_0^A}+\frac{Y_1^A}{Y_0^A}\frac{P_1^A}{Y_1^A}\right)\right]\\ &= E_0\left[\beta\left(\frac{Y_1^A+Y_1^B}{Y_0^A+Y_0^B}\right)^{-1}(1.02)\left(1+\frac{P_1^A}{Y_1^A}\right)\right]\end{aligned}$$

since $\frac{Y_1^A}{Y_0^A}=1.02$.

Social Interactions in Asset Pricing

References

- Becker, G. S., and K. M. Murphy (2000), Social Economics: Market Behavior in a Social Environment, Harvard University Press, Cambridge, MA, Chapters 2 and 9.
- [O] Hong, H., J. D. Kubik, and J. C. Stein (2004), "Social Interaction and Stock-Market Participation," Journal of Finance, 59(1), 137-163.

- Approach 1: There are fads and fashions in asset markets. Introduce "social interactions" in the model to derive Becker-style social demand function.
- Next chapter: Approach 2: Introduce "not rational" investors in addition to "rational" investors. Idea originates from Shiller (1984).

1. A Summary of "Social Forces" (Beckerian Theory)

- Point: Add "social environment" as arguments in utility function.
  - Some investors invest because others do.
- The following set-up is about the consumption of a specific good. (Needs some modification to apply for the financial market.)
- $U^j = U(x^j, y^j; S)$ for consumer j: x^j and y^j are goods; S represents social inflcuences.
  - x^j may be: drugs, crime, going bowling, owning a Rolex watch, voting Democratic, dressing informally at work, keeping a neat lawn, religion, fertility, whether to drive on the right-hand side of the road, ...
  - x^j and S are complements (i.e., if $S \uparrow$, then marginal utility from $x^j \uparrow$).
  - y^j is "everything else".

- S may be also interpreted as "information linkages".
  - One may copy others' choices because he feels that they have superior information (Bikhchandani, Hirshleifer and Welch (1992)).
  - Restaurants, houses, ...
- Now assume
$$S = \frac{1}{N}\sum x^j.$$
N is large enough.
- The result of the utility maximization is a demand function (See Becker and Murphy (2000)):
$$x^j = d^j(e^j, p, S)$$
where e^j is idiosyncratic one (such as income) that affects j. p is price. This function implies
  - Consumer j's demand changes as e^j changes, other things being equal. (This is a usual micro analysis.)
  - Consumer j's demand will increase as p decreases, *other things being equal.*
  - New: Consumer j's demand will increase as other consumers' demand increase (i.e., S increases), other things being equal.
- But the second point here (p's impact on x^j) is not useful. Why? Because p affects x^j directly, but p also affects it indirectly through its impact on S.
- To be more specific, consider how p will affect S (market demand for an average consumer, or "social demand"). Then,
$$\begin{aligned}\frac{dS}{dp} &= \frac{d\left(\frac{1}{N}\sum x^j\right)}{dp} \\ &= \sum \frac{\partial x^j/\partial p}{N} + \sum \frac{\overbrace{(\partial x^j/\partial S)}^{\text{how } S \text{ affects } x} \times \overbrace{(dS/dp)}^{\text{how } p \text{ affects } S}}{N}\end{aligned}$$
- So
$$\frac{dS}{dp}\left(1 - \sum \frac{\partial x^j/\partial S}{N}\right) = \sum \frac{\partial x^j/\partial p}{N}$$

- So

$$\frac{dS}{dp} = \frac{\sum \frac{\partial x^j/\partial p}{N}}{1 - \sum \frac{\partial x^j/\partial S}{N}} = \frac{\sum \frac{\partial x^j/\partial p}{N}}{1 - m}$$

where $m \equiv \sum \frac{\partial x^j/\partial S}{N} > 0$ is the **social multiplier**.

- The numerator, $\sum \frac{\partial x^j/\partial p}{N}$, is the average change in individual demands, other things (including social demand) being equal.
  - If $m = 0$, then there is no social impact. Just $p \uparrow \Leftrightarrow S \downarrow$ as usual.
  - If $0 < m < 1$, then this change, $p \uparrow \Leftrightarrow S \downarrow$, is **magnified**. That is, changes in price that impact most members of a peer group may have very large effects on behavior. (A very small increase in price may substantially decrease the social demand.)
  - If $m > 1$, now $p \uparrow \Leftrightarrow S \uparrow$. What does it mean? This means when $S \uparrow$ for some reason, the social multiplier is so strong that consumers demand more, and hence $p \uparrow$. That is, if $S \uparrow$ by 10%, the demand for average consumer increases by more than 10%, which will further $S \uparrow$. When $m > 1$, the demand is **unstable**.
- If $m < 1$, then $\frac{dS}{dp} < 0$ -> demand function is downward.
- If $m > 1$, then $\frac{dS}{dp} > 0$ -> demand function is upward.
- Figure 9.1 (p. 135) in Becker and Murphy

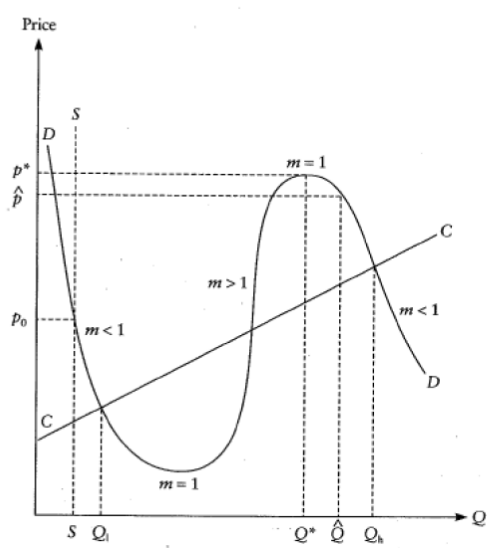


Figure 9.1

- Middle area, $m > 1$: Unstable. An increase in aggreate quantity demanded induces through social interactions a further increase.

- CC is the supply curve.

- Stable equilibria: Q_l and Q_h.

- But a shift between these two involves **an increase in both quantity and quality**.

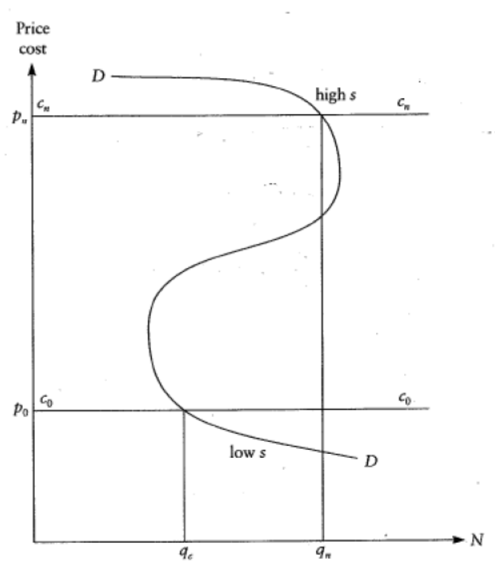
- **Example**: Smoking: Q_h (cigarettes are expensive but peers are smoking a lot) vs. Q_l (cigarettes are cheap but peers don't smoke so you feel less pressure to smoke)

- **Example**: Teenager crime: Q_h (rewards are low, i.e., price is high, but peers commiting many crimes so he/she wants it) vs. Q_l (rewards are high, i.e., prices are low, but peers are not commiting)

- I-phone: i-phone has functions that connect you to other users. If there are more users, you benefit more from using i-phone, and vice versa.

- Now suppose that the demand function has this shape:

- Figure 9.3 (p. 135) in Becker and Murphy



- Figure 9.3

- c_n and c_0 gives stable equilibria.

- If the cost results in an (unstable) equilibrium with $m > 1$, then a small shock will substantially change the price.

2. Example 1: Stock Value as an Argument in the Utility Function

- Owning stocks or other assets may be affected by social influences, too. When neighbours have stocks, I would like more to hold them. Or when neighbours have stocks, I would believe that they have superior information. See, for example, Hong, Kubik and Stein (2004).
- Can we call it "rational"? Is this "psychological"? I am not sure.
- **Stock market interpretation of the above discussion**: If the market is in an equilibrium with $m > 1$, then the price is unstable. Under what conditions $m > 1$ is true? Open question.
- Typical asset-pricing model assumes that the utility for the current period is $u(C_0)$, i.e., it depends only on consumption.
- Assume $u(C_0, a_0, Y_0, P_0)$ where a_0 =amount of asset, Y_0 =endowment, and P_0 =asset price. The whole point is that C_0 is not everything.
- Time-separable version: $u(C_0) + v(a_0, Y_0, P_0)$. Especially,

$$u(C_0) + va_0 \left(\frac{P_0}{Y_0}\right)^2,$$

  where v is now a constant, implying that if asset price (P_0) is higher relative to endowment (Y_0), then the consumer is more eager to get this asset.
- Note: Feel free to apply your own utility functions.
- Note: In equilibrium, $a_0 = 0$ anyway. So the second term disappears.
- Similar in spirit to "money-in-the-utility" model.
- Of course, other functional forms are fine.
- May reflect "social interaction" – When asset prices are high, there is a buzz, and the assets are more exposed to consumers.
- Consider a **risk-free asset**. The utility function follows log preferences:

$$\max_{a_0} \log(Y_0 - a_0 P_0) + va_0 \left(\frac{P_0}{Y_0}\right)^2 + \beta E_0 \left[\log(Y_1 + a_0)\right] + ...$$

- The first-order condition becomes

$$-\frac{P_0}{Y_0 - a_0 P_0} + v\left(\frac{P_0}{Y_0}\right)^2 + \beta E_0\left[\frac{1}{Y_1 + a_0}\right] = 0.$$

- Normalize $Y_0 = 1$ and $Y_1 = 1.07$ w/prob 1/2 and 0.97 w/prob 1/2. Then,

$$-\frac{P_0}{1 - a_0 P_0} + vP_0^2 + \frac{\beta}{2}\left(\frac{1}{1.07 + a_0} + \frac{1}{0.97 + a_0}\right) = 0.$$

$$\text{So } - P_0 + vP_0^2(1 - a_0 P_0) + \frac{\beta}{2}\left(\frac{1}{1.07 + a_0} + \frac{1}{0.97 + a_0}\right)(1 - a_0 P_0) = 0.$$

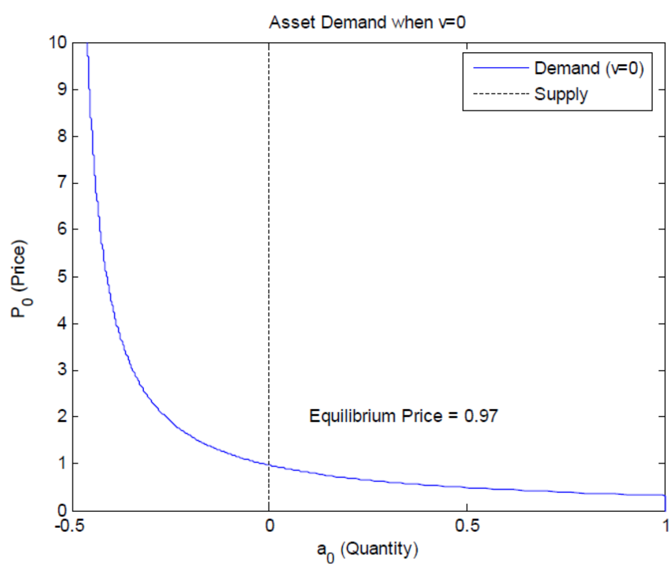
- To be done late: Analytical solutions for this equation? Which pairs of (a_0, P_0) satisfy this equation?

- To do now: Provide numerical solutions. Which solutions are "stable"?

- **Case 1**: If $v = 0$ (no social interaction), then

$$P_0 = \frac{\frac{\beta}{2}\left(\frac{1}{1.07+a_0} + \frac{1}{0.97+a_0}\right)}{\frac{\beta}{2}\left(\frac{1}{1.07+a_0} + \frac{1}{0.97+a_0}\right)a_0 + 1}.$$

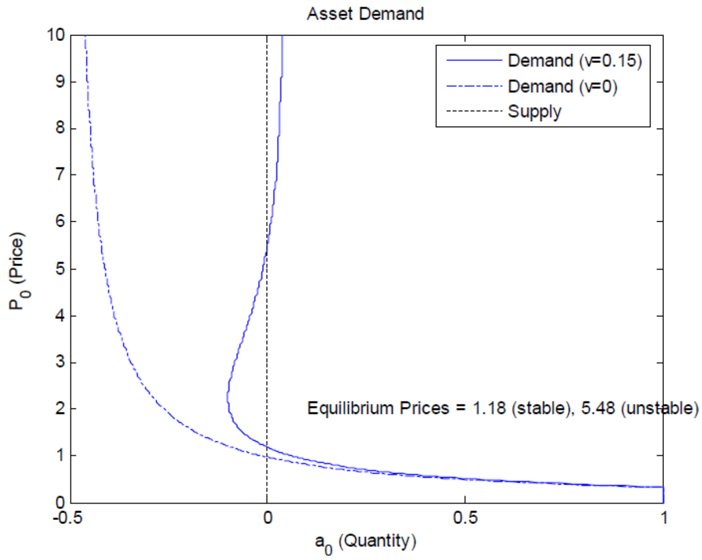
- This first-order condition has the following demand curve (for $\beta = 0.99$):



- In equilibrium in which $a_0 = 0$, we have a familiar result:

$$P_0 = \frac{\beta}{2}\left(\frac{1}{1.07} + \frac{1}{0.97}\right) = 0.97.$$

- **Case 2**: If $v > 0$, then we have multiple equilibria: (for $v = 0.15$):



- In equilibrium in which $a_0 = 0$, the equation becomes

$$P_0^2 - \frac{1}{v}P_0 + \frac{\beta}{2}\frac{1}{v}\left(\frac{1}{1.07} + \frac{1}{0.97}\right) = 0.$$

So there are two equilibrium prices:

$$P_0 = \frac{\frac{1}{v} \pm \sqrt{\frac{1}{v^2} - 4\frac{\beta}{2}\frac{1}{v}\left(\frac{1}{1.07} + \frac{1}{0.97}\right)}}{2} = 1.18 \text{ and } 5.48.$$

- Discussion: When P is low, when P is middle, when P is high, etc., what happens?

- **By Introducing** v:

- (i) Equilibrium price changes – May imply something about equity premium puzzle and risk-free rate puzzle.

- (ii) There are multiple equilibria (although a new one is "unstable"). This may explain excess volatility and the uprising and collapse of a "bubble."

- In addition, a small change on the payoff (or an introduction a new technology) can provide a large change in price. The "new technology" discussion is studied in Pastor and Veronesi (2009).

- Maybe useful to introduce "complementary strategy" argument in Morris and Shin (XXXX).

- Some support from the data? (Investment in this asset, normalized go GDP) = a function of normalized price + β*income + γ*(future perspectives) + epsilon.

3. Example 2: Other's Asset Holding as an Argument in the Utility Function

- $u(C_0) + v\overline{a}_0$ where $\overline{a}_0$ is the average asset holding of "other" investors. (Impose $a_0 = \overline{a}_0$ in equilibrium.)

Not Fully Rational Investors

References

- Shiller, R. J. (1984), "Stock Prices and Social Dynamics," Brooking Papers on Economic Activity, 2:1984, 457-510.

Two related issues

1. Do all investors follow the predictions of the game theory (or economic theory based on "rationality" in general)? If there are some investors who do not follow them, then you should reflect them in your strategy making.

2. Some (dumb?) investors join the asset market only after a hike in asset prices. This may be like a fashion, or just an optimism, or some sort of strategy. But then, "rational" investors will want to react to them to make profits.

Macroeconomic models may have to introduce them. But not a lot of research to this direction.

1. How Will Others Behave?

- Choose one out of 1, 2, ..., 100. Write your name and number.
- Whoever wrote the number closest to 2/3 of the class average wins. (If more than one winner, the award is shared.)
- Nash Equilibrium
  - Your strategy 68, ..., 100 are eliminated. 2/3 of the average cannot exceed 66.67. The maximum number that you are willing to choose is 67.
  - If you know that others will also eliminate 68, ..., 100, then the choices for all players narrow down to 1, 2, ..., 67.
  - But then 2/3 of the average cannot exceed $67 \times 2/3 = 44.67$. Your strategy 46, 47, ... are eliminated.

- If you know that others will also eliminate 46, 47, ..., then the choices for all players narrow down to 1, 2, ..., 45.
- Next step will narrow it down to 1, 2, ..., 30. ($45 * 2/3 = 30$.)
- Next step will narrow it down to 1, 2, ..., 20. ($30 * 2/3 = 20$.)
- Next step will narrow it down to 1, 2, ..., 13. ($20 * 2/3 = 13.33$.)
- Next step will narrow it down to 1, 2, ..., 9. ($13 * 2/3 = 8.67$.)
- Next step will narrow it down to 1, 2, ..., 6. ($9 * 2/3 = 6$.)
- Next step will narrow it down to 1, 2, 3, 4. ($6 * 2/3 = 4$.)
- Next step will narrow it down to 1, 2, 3. ($4 * 2/3 = 2.67$.)
- Next step will narrow it down to 1, 2. ($3 * 2/3 = 2$.)
- Next step will narrow it down to 1. ($2 * 2/3 = 1.33$.)
- Everyone chooses 1 in a Nash equilibrium.

• Lesson: Even though you are "rational" (or a game theorist), if you believe that others are not rational, then you do not choose 1.

• If you believe that some other investors are not completely "rational," your investment decision may be different from the prediction of the asset pricing model (based on the rationality).

• Your belief about the bahaviors of other investors may be important.

2. Schiller's (1984) Example

• $E_t R_t = \delta$ where $R_t = (P_{t+1} - P_t + D_t)/P_t$. This implies

$$E_t[P_{t+1} - P_t + D_t] = \delta P_t$$
$$\text{So } P_t = \frac{E_t[D_t + P_{t+1}]}{1+\delta} = \frac{E_t[D_t]}{1+\delta} + \frac{E_t[P_{t+1}]}{1+\delta}$$

Iterating,

$$P_t = \frac{E_t[D_t]}{1+\delta} + \frac{E_t[D_{t+1}]}{(1+\delta)^2} + ... = \sum_{k=0}^{\infty} \frac{E_t[D_{t+k}]}{(1+\delta)^{k+1}}.$$

• Alternative model: There are rational investors and "not rational" investors.

- Rational investors: Demand for shares:

$$Q_t = \frac{E_t R_t - \rho}{\varphi}$$

This means when $E_t R_t = \rho$, $Q_t = 0$. When $E_t R_t = \rho + \varphi$, $Q_t = 1$.

- Not rational investors: Demand is Y_t in value. Quantity demanded is Y_t / P_t.

- Equilibrium: $Q_t + Y_t / P_t = 1$.

- Hence,

$$E_t R_t = \varphi Q_t + \rho = \varphi(1 - Y_t/P_t) + \rho$$

Since $R_t = (P_{t+1} - P_t + D_t)/P_t$,

$$\begin{aligned} E_t[P_{t+1} - P_t + D_t] &= \varphi(P_t - Y_t) + \rho P_t \\ \text{So } E_t[P_{t+1}] + E_t[D_t] + \varphi Y_t &= (1 + \varphi + \rho) P_t \\ \text{So } P_t &= \frac{E_t[D_t] + \varphi Y_t}{1 + \varphi + \rho} + \frac{E_t[P_{t+1}]}{1 + \varphi + \rho} \\ &= \frac{E_t[D_t] + \varphi Y_t}{1 + \varphi + \rho} + \frac{E_t[D_{t+1}] + \varphi E_t[Y_{t+1}]}{(1 + \varphi + \rho)^2} + \ldots \\ &= \sum_{k=0}^{\infty} \frac{E_t[D_{t+k}] + \varphi E_t[Y_{t+k}]}{(1 + \varphi + \rho)^{k+1}} \end{aligned}$$

- If φ goes to 0, ordinary rational model.

- If φ goes to infinity, then $P_t = Y_t$ (so "not rational" investors determine the price).

- We can further model the situation as:

- Example 1: Y_t is exogenous noise.

- Example 2: Y_t responds to past returns.

- Example 3: Y_t responds to current and lagged dividends.

- ...

3. Applications to Asset-Pricing Models

- An endowment economy. Investors with identical endowments.

- The asset is the wealth portfolio (providing endowment).

- There are two investors, types A and B.
- Type A is a typical rational investor, maximizing at period t:

$$E_t\left[\sum_{\tau=0}^{\infty}\beta^{\tau}u(C_{t+\tau}^{A})\right],\quad 0<\beta<1.$$

- Constraints:

$$\begin{aligned} C_t &= Y_t - a_t^A P_t, \\ C_{t+1} &= Y_{t+1} + a_t^A X_{t+1}, \\ C_{t+2} &= Y_{t+2}, \ldots \end{aligned}$$

- Type B is not a typical rational investor.
  - $a_t^B = \bar{b}(r_t - \bar{r})$: "Adaptive" investor. He observes the realized return and passively determine how much to invest.
  - Or, choosing a_t^B to maximize

    $$u(C_t^B, a_t^B P_t) + E_t\left[\sum_{\tau=1}^{\infty}\beta^{\tau}u(C_{t+\tau}^{B})\right],$$

    subject to the same constraint. (This is more "Becker-style".) He likes to invest more when prices are higher.
  - Or, try some other set-ups.
- Either way, the equilibrium is $a_t^A + a_t^B = 0$.
- So what is the impact of the type-B investor?

Defaults and Punishments

References

- Dubey, P., J. Geanakoplos and M. Shubik (2005), "Default and Punishment in General Equilibrium," Econometrica, 73(1), 1-37.

- Reference: Example 1 (p. 18) in Dubey, Geanakoplos and Shubik (2005), "Default and Punishment in General Equilibrium," *Econometrica*.
- We assumed that *every* agent keeps the promise.
- In reality, many agents "default". We want to introduce it in our general equil. model. Look for implications (?).

1. Environment

- Simplest possible Arrow-Debreu environment.
- 2 periods. 2 states. 2 consumers.
- We are at period 0.
-

Period 1's possible states:	State #1	State #2
Consumer #1's endowment	1	0
Consumer #2's endowment	0	1
Agent #1's assigned probability	2/3	1/3
Agent #2's assigned probability	1/3	2/3
Asset #1's payment	1	0
Asset #2's payment	0	1

- "The economy is more optimistic for me!"
- **Problem 1**: Arrow-Debreu Allocation under "Complete" Market

  No defaults introduced yet!
- Preferences for both consumers: $u(c) = \log c$.
- Consumer 1 solves

$$\max_{c_1^1, c_2^1} \beta \left(\frac{2}{3} \log(c_1^1) + \frac{1}{3} \log(c_2^1) \right)$$

(where: c_j^i: Consumer i's consumption at state j)

- s.t.

$$\begin{aligned} q_1 c_1^1 + q_2 c_2^1 &\le q_1 e_1^1 + q_2 e_2^1 \\ &= q_1 \end{aligned}$$

- $c_1^1 = 1 \times a_1^1 + 0 \times a_2^1$, $c_2^1 = 1 \times a_2^1 + 0 \times a_2^2$
- [Add figure]
- (where: q_j: Period-0 price of Asset $\#j$, Period-0 price of an Arrow-Debreu security paying 1 at state j at period 1 and 0 otherwise, equivalent to $q_1^0(s = j)$ in our previous notation.)
- Consumer 2 solves a symmetric problem:

$$\max_{c_1^2, c_2^2} \beta \left(\frac{1}{3} \log(x_1^2) + \frac{2}{3} \log(x_2^2) \right)$$

s.t.

$$\begin{aligned} q_1 c_1^2 + q_2 c_2^2 &\le q_1 e_1^2 + q_2 e_2^2 \\ &= q_2 \end{aligned}$$

- Market clearing:

$$\begin{aligned} c_1^1 + c_1^2 = e_1^1 + e_1^2 = 1, \\ c_2^1 + c_2^2 = e_2^1 + e_2^2 = 1. \end{aligned}$$

- **Solution 1**:
- Lagrangian for Consumer 1:

$$L^1 = \frac{2}{3} \log(c_1^1) + \frac{1}{3} \log(c_2^1) + \lambda^1 \left(q_1 - q_1 c_1^1 - q_2 c_2^1 \right)$$

- The FOCs for Agent 1 are

$$\begin{aligned} \frac{2}{3} \frac{1}{c_1^1} &= \lambda^1 q_1, \\ \frac{1}{3} \frac{1}{c_2^1} &= \lambda^1 q_2, \\ q_1 c_1^1 + q_2 c_2^1 &= q_1. \text{ (constraint itself)} \end{aligned}$$

Eliminating λ^1,

$$2\frac{c_2^1}{c_1^1} = \frac{q_1}{q_2}.$$

The constraint becomes

$$q_2 c_2^1 = (1 - c_1^1) q_1.$$

$$\text{So } \frac{c_2^1}{1 - c_1^1} = \frac{q_1}{q_2}.$$

Finally

$$2\frac{c_2^1}{c_1^1} = \frac{c_2^1}{1 - c_1^1}.$$

$$\text{So } 2(1 - c_1^1) = c_1^1.$$

$$\text{So } c_1^1 = \frac{2}{3}.$$

(What's wrong if $c_2^1 = 0$? Then corner solution. That is, log(0) in the utility.)

- The FOCs for Agent 2 will similarly give $c_2^2 = \frac{2}{3}$.
- From market clearing, $c_2^1 = c_1^2 = \frac{1}{3}$.
- Finally, the contraint gives $\frac{q_1}{q_2} = 1$.
- **Solution 1** (Tricky Version):
- For Agent 1:
  - If State #1 occurs, good! If State #2 occurs, bad.
  - It is obvious that Agent 1 sells Asset #1 and buys Asset #2.
  - The only problem is how many units of Asset #1 Agent 1 will sell (and how many units of Asset #2 Agent 1 will buy)
- Agent 1's problem:

$$\max \frac{2}{3}\log(1 - D) + \frac{1}{3}\log\theta$$

(D: Agent 1's **delivery** to Agent 2, if State #1 occurs at period 1.

θ: Agent 2's delivery to Agent 1, if State 2 occurs at period 1.)

- Subject to the budget constraint:

$$\underbrace{q_1 D}_{\text{Agent 1's sales}} = \underbrace{q_2 \theta}_{\text{Agent 1's purchases}}.$$

- This is a symmetric problem, so WE KNOW $q_1 = q_2$.
- Eventually the problem is

$$\max \frac{2}{3} \log(1-D) + \frac{1}{3} \log D$$

- FOC:

$$\frac{2}{3} \frac{-1}{1-D} + \frac{1}{3} \frac{1}{D} = 0.$$

So

$$2D = 1 - D.$$
$$\text{So } D = \frac{1}{3}.$$

2. Default and Punishment

- New Set-up: You don't have to provide a "full" delivery.
- Example:

<u>You sold</u>	<u>Now state #1 occured. You deliver</u>
1 unit of Asset #1	1 unit of physical good
(You are obliged to to	(in a complete market)
deliver 1 unit of consumption	But you may default now!
good if state #1 occurs.)	$\rightarrow$ 2/3 units of physical good
	Default on 1/3 unit. But receive

punishment.

- **Problem 2**: Agent 1's problem: Maximize

$$\frac{2}{3}\left[\log(\underbrace{1 - D_{11}^1}_{=c_1^1}) - \lambda(\underbrace{\varphi_1^1 - D_{11}^1}_{\text{amount defaulted}})^+\right] + \frac{1}{3}\left[\log(\underbrace{\theta_2^1 \times K_{22}}_{=c_2^1})\right]$$

- φ^i_j (*varphi*): Agent i's sales of Asset #j
  (i.e., if $\varphi^1_1 = \frac{1}{3}$, then Agent 1 sold $\frac{1}{3}$ units of Asset #1. So he promised to deliver $\frac{1}{3}$ units of physical goods if state 1 occurs and 0 unit if state 2 occurs.)
- D^i_{sj}: Agent i's delivery of physical goods, at state s, regarding Asset #j
  (i.e., if $D^1_{11} = \frac{1}{9}$, then Agent 1 delivered only $\frac{1}{9}$ units of physical goods after state 1 has occured, regarding Asset #1. This means he defauted on $\frac{2}{9}$ units.)
- Notice a plus sign in $(\underbrace{\varphi^1_1 - D^1_{11}}_{\text{amount defaulted}})^+$. If $\varphi^1_1 - D^1_{11}$ is negative, this agent is paying even more than he promised. This will not happen!
- λ: Default penalty per unit of defaulted physical goods (e.g., legal punishment)
- θ^i_j: Agent i's purchases of Asset #j
  (i.e., if $\theta^1_2 = \frac{2}{3}$, then Agent 1 bought $\frac{2}{3}$ units of Asset #2. If Asset #2 doesn't default at all, he is delivered $\frac{2}{3}$ units of physical goods if state 2 occurs.)
- K_{sj}: Asset #j's delivery rate at state s
  (i.e., if $K_{sj} = 0.5$, then only 50% of promised $\frac{2}{3}$ units are delivered.)

• Subject to the budget constraint:

$$\underbrace{q_1\varphi^1_1}_{\text{Agent 1's sales}} = \underbrace{q_2\theta^1_2}_{\text{Agent 1's purchases}}.$$

• This is a symmetric problem, so WE KNOW $q_1 = q_2$.

• Eventually the unconstrained problem is

$$\max_{D,\,\varphi} \frac{2}{3}\left[\log(1-D) - \lambda(\varphi - D)^+\right] + \frac{1}{3}\left[\log(\varphi \times \underbrace{K}_{\text{not the control of Agent 1}})\right]$$

where I omitted all superscripts and subscripts since there are no confusions.

• Difficult because "+"!

• So consumer chooses φ at period 0 and then D at period 1.

• STEP 1: Let's say φ is determined, and see how Agent 1 chooses D.

• If $\varphi = 0$, then obviously $D = 0$.

- If $0 < \varphi < 1$, then D is between 0 and φ. (The agent doesn't gain at all by delivering more than what is promised!)

```
      |
      |
      |               φ        1
      +————————————————#########——— D
      |       +      #            ⟹ −λ(φ − D)⁺
      |          #     +
      |       #   *******      +
      |    #   *                 +
  −λφ#*                           +
      |                            + ⟹ log(1 − D)
```

- Solutions are one of the following three:

  (i) $D^* = 0$

  (ii) D^* is a solution to $\max_D \log(1-D) - \lambda(\varphi - D)$ and this D is between 0 and φ.

  (iii) $D^* = \varphi$.

- Let's discuss these solutions in detail.

- Consider: $\max_{D>0} \log(1-D) - \lambda(\varphi - D)$. (without "+".)

- The first derivative is $\frac{-1}{1-D} + \lambda$.

- (a) If $\frac{-1}{1-D} + \lambda = 0$ for some $D < 0$, then it is obvious that $D^* = 0$. (Show by figure.)

  Solution: $D = 1 - \frac{1}{\lambda}$

  $\Leftrightarrow$ If $D = 1 - \frac{1}{\lambda} < 0$, then it is obvious that $D^* = 0$.

  $\Leftrightarrow$ $\boxed{\text{If } \lambda < 1, \text{ then } D^* = 0.}$

- (b) If $\frac{-1}{1-D} + \lambda = 0$ for some $0 < D < \varphi$, then it is obvious that $D^* = 1 - \frac{1}{\lambda}$. (Show by figure.)

  Solution: $D = 1 - \frac{1}{\lambda}$

  $\Leftrightarrow$ If $0 < 1 - \frac{1}{\lambda} < \varphi$, then it is obvious that $D^* = 1 - \frac{1}{\lambda}$.

  $\Leftrightarrow$ $\boxed{\text{If } 1 < \lambda < \frac{1}{1-\varphi}, \text{ then } D^* = 1 - \frac{1}{\lambda}.}$

- (c) If $\frac{-1}{1-D} + \lambda = 0$ for some $D > \varphi$, then it is obvious that $D^* = \varphi$. (Show by figure.)

  $\Leftrightarrow$ If $\varphi < 1 - \frac{1}{\lambda}$, then it is obvious that $D^* = \varphi$.

  $\Leftrightarrow$ $\boxed{\text{If } \frac{1}{1-\varphi} < \lambda, \text{ then } D^* = \varphi. \text{ (Full delivery! No Default!)}}$

- STEP 2: What is φ?

- (a) If $\lambda < 1$, then $D^* = 0$. By symmetricity, $K = 0$. Nobody delivers anything. Nobody purchases any assets. $\varphi = 0$.

- (b) For λ and φ satisfying $1 < \lambda < \frac{1}{1-\varphi}$, we have $D^* = 1 - \frac{1}{\lambda}$. The problem is

$$\max_{\varphi} \frac{2}{3}\left[\log(1 - 1 + \frac{1}{\lambda}) - \lambda(\varphi - 1 + \frac{1}{\lambda})\right] + \frac{1}{3}\left[\log(\varphi \times K)\right].$$

  FOC is

$$-\frac{2}{3}\lambda + \frac{1}{3}\frac{1}{\varphi} = 0.$$
$$\text{So } 2\lambda\varphi = 1.$$
$$\text{So } \varphi = \frac{1}{2\lambda}.$$

  But this is meaningful if it satisfies $1 < \lambda < \frac{1}{1-\varphi}$. That is,

$$\lambda < \frac{1}{1 - \frac{1}{2\lambda}}.$$
$$\text{So } \lambda < \frac{1}{\frac{2\lambda-1}{2\lambda}}.$$
$$\text{So } \lambda < \frac{2\lambda}{2\lambda - 1}.$$
$$\text{So } 2\lambda - 1 < 2.$$
$$\text{So } \lambda < \frac{3}{2}.$$

- Summary: If $1 < \lambda < \frac{3}{2}$, then $D^* = 1 - \frac{1}{\lambda}$ and $\varphi^* = \frac{1}{2\lambda}$.

- (c) For λ and φ satisfying $\frac{1}{1-\varphi} < \lambda$, we have $D^* = \varphi$. The problem is

$$\max_{\varphi} \frac{2}{3}\left[\log(1 - \varphi)\right] + \frac{1}{3}\left[\log(\varphi \times K)\right].$$

FOC is

$$\frac{2}{3}\frac{-1}{1-\varphi}+\frac{1}{3}\frac{1}{\varphi}=0.$$

$$\text{So } \varphi=\frac{1}{3}.$$

- Same as the perfect market!
- But this is meaningful if it satisfies $\frac{1}{1-\varphi}<\lambda$. That is,

$$\frac{1}{1-\frac{1}{3}}<\lambda.$$

$$\text{So } \lambda>\frac{3}{2}.$$

- Conclusion:
-

	Punishment	D^*	φ^*	K	
$\lambda>\frac{3}{2}$	High	$\frac{1}{3}$	$\frac{1}{3}$	1	Same as Perfect Market!
$1<\lambda<\frac{3}{2}$	Medium	$1-\frac{1}{\lambda}$	$\frac{1}{2\lambda}$	$\frac{1-\frac{1}{\lambda}}{\frac{1}{2\lambda}}=2(\lambda-1)$	
$\lambda<1$	Low	0	0	0	No Market

EXERCISES

1. *(Dubey, Geanakoplos and Shubik (Econometrica, 2005))* In class, we considered an economy with 2 periods, 2 states, and 2 consumers with the following environment:

Table 1.

Period 1's possible states:	State #1	State #2
Agent A's endowment	1	0
Agent B's endowment	0	1
Agent A's assigned probability	$\frac{2}{3}$	$\frac{1}{3}$
Agent B's assigned probability	$\frac{1}{3}$	$\frac{2}{3}$
Asset #1's payment	1	0
Asset #2's payment	0	1

Each agent has log preferences on her consumption. If an agent defaults, she has a utility punishment of λ for each unit of physical good defaulted. Our result was as follows:

Table 2.

	Punishment	φ^*	D^*	K^*	
$\lambda > \frac{3}{2}$	High	$\frac{1}{3}$	$\frac{1}{3}$	1	Same as the complete market
$1 < \lambda < \frac{3}{2}$	Medium	$\frac{1}{2\lambda}$	$1 - \frac{1}{\lambda}$	$2(\lambda - 1)$	
$\lambda < 1$	Low	0	0	0	No market

Here, φ^* is Agent A's sales of Asset #1, D^* is Agent A's delivery regarding Asset #1 at State #1, and $K^* = D^*/\varphi^*$ is a delivery rate of Asset #1. By symmetricity, these are also applied to Asset #2 sold by Agent B.

In this problem, we consider the same environment, but now there is a third asset, named Asset #0. This asset promises to pay 1 unit of physical good regardless of the state at period 1. (If fully delivered at any state, this asset is risk-free.) That is, add the following to Table 1:

Period 1's possible states:	State #1	State #2
Asset #0's payment	1	1

Our goal is to create a version of Table 2 in this updated environment. Formally, Agent A's problem is to maximize

$$\begin{aligned}
&\frac{2}{3}[\log(1 - D^A_{10} - D^A_{11} - D^A_{12} + \theta^A_0 \times K^B_{10} + \theta^A_1 \times K^B_{11} + \theta^A_2 \times K^B_{12}) \\
&- \lambda(1 \times \varphi^A_0 - D^A_{10})^+ - \lambda(1 \times \varphi^A_1 - D^A_{11})^+ - \lambda(0 \times \varphi^A_2 - D^A_{12})^+] \\
&+ \frac{1}{3}[\log(-D^A_{20} - D^A_{21} - D^A_{22} + \theta^A_0 \times K^B_{20} + \theta^A_1 \times K^B_{21} + \theta^A_2 \times K^B_{22}) \\
&- \lambda(1 \times \varphi^A_0 - D^A_{20})^+ - \lambda(0 \times \varphi^A_1 - D^A_{21})^+ - \lambda(1 \times \varphi^A_2 - D^A_{22})^+]
\end{aligned}$$

subject to

$$q_0\varphi^A_0 + q_1\varphi^A_1 + q_2\varphi^A_2 = q_0\theta^A_0 + q_1\theta^A_1 + q_2\theta^A_2.$$

Here,

- φ^i_j: Agent i's sales of Asset $\#j$
- D^i_{sj}: Agent i's delivery of physical goods, at state s, regarding Asset $\#j$
- λ: Default penalty per unit of defaulted physical good (e.g., legal punishment)
- θ^i_j: Agent i's purchases of Asset $\#j$
- K^i_{sj}: Asset $\#j$'s delivery rate at state s, determined by Agent ι
- q_j: Asset $\#j$'s price (at period 0)

(a) Write down Agent B's problem.

(b) Since Agent B's problem is symmetric to Agent A's, we focus on Agent A's. Consider Agent A. If State #2 occurs, she doesn't have any endowment, so she has nothing to deliver! Hence, for example, $D^A_{20} = 0$. Similarly, Agent B has nothing to deliver if State #1 occurs. This means, for example, $K^B_{10} = 0$. Also, Asset #2 does not promise to deliver anything at State #1. Hence, $D^A_{12} = 0$. Find all such conditions and simplify Agent A's problem.

Answer: *The parts marked with* $\underbrace{\quad}$ *are obviously zero:*

$$\begin{aligned}
&\frac{2}{3}[\log(1 - D^A_{10} - D^A_{11} - \underbrace{D^A_{12}} + \theta^A_0 \times \underbrace{K^B_{10}} + \underbrace{\theta^A_1} \times \underbrace{K^B_{11}} + \theta^A_2 \times \underbrace{K^B_{12}}) \\
&- \lambda(\varphi^A_0 - D^A_{10})^+ - \lambda(\varphi^A_1 - D^A_{11})^+ - \lambda(0\ \underbrace{\varphi^A_2} - \underbrace{D^A_{12}})^+] \\
&+ \frac{1}{3}[\log(-\underbrace{D^A_{20}} - \underbrace{D^A_{21}} - \underbrace{D^A_{22}} + \theta^A_0 \times K^B_{20} + \underbrace{\theta^A_1} \times \underbrace{K^B_{21}} + \theta^A_2 \times K^B_{22}) \\
&- \lambda(\varphi^A_0 - \underbrace{D^A_{20}})^+ - \lambda(0\varphi^A_1 - \underbrace{D^A_{21}})^+ - \lambda(\underbrace{\varphi^A_2} - \underbrace{D^A_{22}})^+]
\end{aligned}$$

s.t.

$$q_0\varphi^A_0 + q_1\varphi^A_1 + q_2\underbrace{\varphi^A_2} = q_0\theta^A_0 + q_1\underbrace{\theta^A_1} + q_2\theta^A_2.$$

The problem now is

$$\frac{2}{3}[\log(1 - D_{10}^A - D_{11}^A) - \lambda(\varphi_0^A - D_{10}^A)^+ - \lambda(\varphi_1^A - D_{11}^A)^+]$$
$$+ \frac{1}{3}[\log(\theta_0^A \times K_{20}^B + \theta_2^A \times K_{22}^B) - \lambda\varphi_0^A]$$

s.t.

$$q_0\varphi_0^A + q_1\varphi_1^A = q_0\theta_0^A + q_2\theta_2^A.$$

(c) Throughout this problem, we look for a symmetric equilibrium. For example, we assume $\varphi_0^A = \theta_0^A$ so that A's sales of Asset #0 to B are the same as A's purchases of Asset #0 from B. Also, by symmetry, we will have $q_1 = q_2$. Show that under these two conditions, $\varphi_0^A = \theta_0^A$ and $q_1 = q_2$, Agent A's problem becomes the following unconstrained maximization problem:

$$\max_{D_{10}^A, D_{11}^A, \varphi_0^A, \varphi_1^A} \frac{2}{3}[\log(1 - D_{10}^A - D_{11}^A) - \lambda(\varphi_0^A - D_{10}^A)^+ - \lambda(\varphi_1^A - D_{11}^A)^+]$$
$$+ \frac{1}{3}[\log(\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B) - \lambda\varphi_0^A]$$

Answer: $\varphi_0^A = \theta_0^A$ *and* $\varphi_1^A = \theta_2^A$ *(direct from* $q_1 = q_2$*).*

(d) Now we solve the problem. By set-up, at period 0, Agent A should determine φ_0^A and φ_1^A. At period 1, Agent A should determine D_{10}^A and D_{11}^A. We first solve for D_{10}^A and D_{11}^A, given φ_0^A and φ_1^A. In this case, the term $\frac{1}{3}[\log(\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B) - \lambda\varphi_0^A]$ disappears because there are neither D_{10}^A nor D_{11}^A. It is easy to see that the problem now is

$$\max_{0 \le D_{10}^A \le \varphi_0^A,\ 0 \le D_{11}^A \le \varphi_1^A} \frac{2}{3}[\log(1 - D_{10}^A - D_{11}^A) - \lambda(\varphi_0^A + \varphi_1^A - D_{10}^A - D_{11}^A)]$$

Notice that this problem adds constaints, $0 \le D_{10}^A \le \varphi_0^A$ and $0 \le D_{11}^A \le \varphi_1^A$. Of course, if these are satisfied, then we don't need "plus" signs as in $(\varphi_0^A - D_{10}^A)^+$ or $(\varphi_1^A - D_{11}^A)^+$. Complete the following table:

Table 3.

		Punishment	Optimal $D_{10}^A + D_{11}^A$
Case 1	$\lambda >?$	High	? (full delivery)
Case 2	$? < \lambda <?$	Medium	? (partial delivery)
Case 3	$\lambda <?$	Low	0 (no delivery)

(The question marks (?'s) in the second column should be replaced by numbers or functions of φ_0^A and φ_1^A.)

Answer: *The objective is a function only of* $D_{10}^A + D_{11}^A$. *(*D_{10}^A *or* D_{11}^A *does not act separately. Only the sum is important!) The FOC wrt* $D_{10}^A + D_{11}^A$ *is*

$$\frac{-1}{1 - D_{10}^A - D_{11}^A} + \lambda = 0,$$

implying that

$$D_{10}^A + D_{11}^A = 1 - \frac{1}{\lambda}.$$

But we have constaints, $0 \le D_{10}^A \le \varphi_0^A$ *and* $0 \le D_{11}^A \le \varphi_1^A$. *In our context, the constraint is* $0 \le D_{10}^A + D_{11}^A \le \varphi_0^A + \varphi_1^A$. *This gives the following three cases:*
Case 2: If $0 < D_{10}^A + D_{11}^A = 1 - \frac{1}{\lambda} < \varphi_0^A + \varphi_1^A$, *then no problem!* $D_{10}^A + D_{11}^A = 1 - \frac{1}{\lambda}$ *is a solution, and* $1 < \lambda < \frac{1}{1-\varphi_0^A-\varphi_1^A}$.
The other two cases are corner solutions:
Case 3: If $D_{10}^A + D_{11}^A = 1 - \frac{1}{\lambda} < 0$, *then* $D_{10}^A + D_{11}^A = 0$ *(no delivery). The condition* $1 - \frac{1}{\lambda} < 0$ *is equivalent to* $\lambda < 1$.
Case 1: If $D_{10}^A + D_{11}^A = 1 - \frac{1}{\lambda} > \varphi_0^A + \varphi_1^A$, *then* $D_{10}^A + D_{11}^A = \varphi_0^A + \varphi_1^A$ *(full delivery). The condition* $1 - \frac{1}{\lambda} > \varphi_0^A + \varphi_1^A$ *is equivalent to* $\lambda > \frac{1}{1-\varphi_0^A-\varphi_1^A}$.

		Punishment	Optimal $D_{10}^A + D_{11}^A$
Case 1	$\lambda > \frac{1}{1-\varphi_0^A-\varphi_1^A}$	High	$\varphi_0^A + \varphi_1^A$ (full delivery)
Case 2	$1 < \lambda < \frac{1}{1-\varphi_0^A-\varphi_1^A}$	Medium	$1 - \frac{1}{\lambda}$ (partial delivery)
Case 3	$\lambda < 1$	Low	0 (no delivery)

(e) Now we go back to our original problem to solve for φ_0^A and φ_1^A that maximize Agent A's expected utility. Our strategy is as follows. First, we consider each of Case 1, Case 2 and Case 3. Assume the condition about λ in each case (specified in the second column of Table 3) is satisfied so that we can use our solutions on $D_{10}^A + D_{11}^A$. Then, solve for φ_0^A and φ_1^A. Finally, check whether the assumed condition on λ is in fact satisfied under these solutions.

We start with Case 3 (no delivery). "Argue" that $\varphi_0^A = \varphi_1^A = 0$. (For this case, we don't even have to obtain the first-order condition.)

Answer: *Agent A does not deliver any goods. By symmetry, Agent B does not deliver any goods. They all know that each other will not deliver any goods. Nobody will buy assets, so there is no asset market.*

(f) Now consider Case 1 (full delivery). Show that the problem now is

$$\max_{\varphi_0^A, \varphi_1^A} \frac{2}{3}[\log(1 - \varphi_0^A - \varphi_1^A)] + \frac{1}{3}[\log(\varphi_0^A + \varphi_1^A) - \lambda\varphi_0^A].$$

Answer: *Case 1 implies* $D^A_{10} = \varphi^A_0$ *and* $D^A_{11} = \varphi^A_1$*. So*

$$\max_{\varphi^A_0, \varphi^A_1} \frac{2}{3}[\log(1 - \varphi^A_0 - \varphi^A_1)] + \frac{1}{3}[\log(\varphi^A_0 \times K^B_{20} + \varphi^A_1 \times K^B_{22}) - \lambda\varphi^A_0]$$

By symmetricity, it is also a full delivery for Agent B: $D^B_{20} = \varphi^B_0$ *and* $D^B_{22} = \varphi^A_2$*. So we know* $K^B_{20} = 1$ *and* $K^B_{22} = 1$.

(g) In this type of optimization problem, we take the first derivatives with respect to φ^A_0 and φ^A_1 in the objective function and set them equal to zero. This doesn't work in our problem. To see this quickly, notice that the first-order condition with respect to φ^A_0 is

$$(1) \qquad \frac{2}{3}\frac{-1}{1 - \varphi^A_0 - \varphi^A_1} + \frac{1}{3}\left(\frac{1}{\varphi^A_0 + \varphi^A_1}\right) - \frac{1}{3}\lambda = 0.$$

The first-order condition with respect to φ^A_1 is

$$(2) \qquad \frac{2}{3}\frac{-1}{1 - \varphi^A_0 - \varphi^A_1} + \frac{1}{3}\left(\frac{1}{\varphi^A_0 + \varphi^A_1}\right) = 0.$$

Can you find the solutions satisfying (1) and (2)? Obviously no. What is going on? In Case 1, the punishment (λ) is high so Agent A never defaults on Asset #1. But this doesn't mean that Agent A will never default on Asset #0. Why? If State #2 is realized, Agent A doesn't have any endowment to deliver, so she will default no matter how large λ is. (We said "full delivery", but that applies to the cases in which you do have enough endowments to deliver! I agree that this is a confusing term.) Agent B confronts the same situation. She will not default on Asset #2. But she will default on Asset #0 if State #1 occurs. Then, Asset #0 is not needed in the market. To see this, suppose there is a third party who wants to buy a risk-free asset. She can simply buy one unit of Asset #1 from Agent A and one unit of Asset #2 from Agent B, both of which will never be defaulted. Why does she bother to buy Asset #0 which will be defaulted? In short, the market works perfectly without Asset #0.

If you do not see this intuition very clearly, then another way to explain the same logic is to look at the above two equations, (1) and (2). We go back to ECON 101 and think. Suppose there are two choices. If you have an additional $1, you spend it on the choice that gives a higher marginal utility. The left-hand sides of (1) and (2) are the (expected) marginal utilities on two choices, "selling Asset #0" and "selling Asset #1". It is obvious that the left-hand side of (2) is higher – "selling Asset #1" is superior for Agent A. Asset #0 will not be issued by Agent A.

Now solve the problem. So what are the levels of φ_0^A and φ_1^A that maximize Agent #1's expected utility? Under what condition on λ is it true? (That is, write the third column of Table 3 without φ_0^A and φ_1^A.)

Answer: *The problem, again, is*

$$\max_{\varphi_1^A} \frac{2}{3}[\log(1-\varphi_1^A)] + \frac{1}{3}[\log(\varphi_1^A)].$$

This is exciting because it is exactly the same as the complete market. $\varphi_1^A = \frac{1}{3}$. *Notice that for this to be the solution,* $1 - \frac{1}{\lambda} > \varphi_0^A + \varphi_1^A$. *So*

$$1 - \frac{1}{\lambda} > \frac{1}{3}.$$
$$\text{So } \lambda > \frac{3}{2}.$$

(h) Now consider Case 2 (partial delivery). Show that the problem now is

$$\max_{\varphi_0^A,\varphi_1^A} \frac{2}{3}\left[\log\left(\frac{1}{\lambda}\right) - \lambda\left(\varphi_0^A + \varphi_1^A - 1 + \frac{1}{\lambda}\right)\right] + \frac{1}{3}[\log(\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B) - \lambda\varphi_0^A].$$

Answer: *Case 2:* $0 < 1 - \frac{1}{\lambda} < \varphi_0^A + \varphi_1^A$ *(so* $1 < \lambda < \frac{1}{1-\varphi_0^A-\varphi_1^A}$*) and* $D_{10}^A + D_{11}^A = 1 - \frac{1}{\lambda}$. *Clear from the original problem.*

(i) The two first-order conditions, obtained in an usual way, are

$$(3) \qquad -\frac{2}{3}\lambda + \frac{1}{3}\left(\frac{K_{20}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B} - \lambda\right) = 0.$$
$$(4) \qquad -\frac{2}{3}\lambda + \frac{1}{3}\frac{K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B} = 0.$$

This is an equation system of two equations and two unknowns, but again, there are no solutions for φ_0^A and φ_1^A. If the left-hand side of (3) is greater than the left-hand side of (4), then Agent A sells Asset #0 only. Show that if $1 < \lambda < \frac{4}{3}$, then there is a symmetric equilibrium in which $\varphi_0^A = \frac{1}{3\lambda}$ and $\varphi_1^A = 0$. Also, obtain the solutions for D_{10}^A, D_{11}^A, and K_{20}^B in this case.

Answer: *Agent A sells Asset #0 only, so the problem is*

$$\max_{\varphi_0^A} \frac{2}{3}\left[\log\left(\frac{1}{\lambda}\right) - \lambda\left(\varphi_0^A - 1 + \frac{1}{\lambda}\right)\right] + \frac{1}{3}[\log(\varphi_0^A \times K_{20}^B) - \lambda\varphi_0^A].$$

The first-order condition is

$$-\frac{2}{3}\lambda + \frac{1}{3}\frac{1}{\varphi_0^A} - \frac{1}{3}\lambda = 0.$$

$$\text{So } \frac{1}{\varphi_0^A} = 3\lambda.$$

$$\text{So } \varphi_0^A = \frac{1}{3\lambda}.$$

But this is the solution under the following two conditions:

i. $1 < \lambda < \frac{1}{1-\varphi_0^A-\varphi_1^A}$*: This is the condition for Case 2 itself. It becomes* $1 < \lambda < \frac{1}{1-\frac{1}{3\lambda}-0} = \frac{3\lambda}{3\lambda-1}$*. This is equivalent to* $\boxed{1 < \lambda < \frac{4}{3}}$.

ii. *The left-hand side of (3) is greater than the left-hand side of (4):*

$$-\frac{2}{3}\lambda + \frac{1}{3}\left(\frac{K_{20}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B} - \lambda\right) > -\frac{2}{3}\lambda + \frac{1}{3}\frac{K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B}.$$

This condition is equivalent to

$$\frac{K_{20}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B} - \lambda > \frac{K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B}$$

$$\text{or, } \lambda < \frac{K_{20}^B - K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B}.$$

By definition, $K_{20}^B = D_{20}^B/\varphi_0^B$ *and* $K_{22}^B = D_{22}^B/\varphi_2^B$*. By symmetricity,*

$$K_{20}^B = D_{20}^B/\varphi_0^B = D_{10}^A/\varphi_0^A.$$
$$K_{22}^B = D_{22}^B/\varphi_2^B = D_{11}^A/\varphi_1^A = 0.$$

Inserting these, the condition becomes

$$\lambda < \frac{D_{10}^A/\varphi_0^A}{D_{10}^A} = \frac{1}{\varphi_0^A} = \frac{1}{\frac{1}{3\lambda}} = 3\lambda$$

which is always satisfied.

And in this case, $D_{11}^A = 0$ *(since* $\varphi_1^A = 0$*),* $D_{10}^A = 1 - \frac{1}{\lambda}$*, and* $K_{10}^A = K_{20}^B = \frac{1-\frac{1}{\lambda}}{\frac{1}{3\lambda}} = 3(\lambda - 1)$.

(j) If the left-hand side of (4) is greater than the left-hand side of (3), then Agent A sells Asset #1 only. Show that if $1 < \lambda < \frac{3}{2}$, then there is a symmetric equilibrium in which $\varphi_0^A = 0$ and $\varphi_1^A = \frac{1}{2\lambda}$. Also, obtain the solutions for D_{10}^A, D_{11}^A, and K_{22}^B in this case.

Answer: *Agent A sells Asset #1 only, so the problem is*

$$\max_{\varphi_1^A} \frac{2}{3}\left[\log\left(\frac{1}{\lambda}\right) - \lambda\left(\varphi_1^A - 1 + \frac{1}{\lambda}\right)\right] + \frac{1}{3}[\log(\varphi_1^A \times K_{22}^B)].$$

The first-order condition is

$$-\frac{2}{3}\lambda + \frac{1}{3}\frac{1}{\varphi_1^A} = 0.$$
$$\text{So } \frac{1}{\varphi_1^A} = 2\lambda.$$
$$\text{So } \varphi_1^A = \frac{1}{2\lambda}.$$

But this is the solution under the following two conditions:

i. $1 < \lambda < \frac{1}{1-\varphi_0^A-\varphi_1^A}$*: This is the condition for Case 2 itself. It becomes* $1 < \lambda < \frac{1}{1-\frac{1}{2\lambda}-0} = \frac{2\lambda}{2\lambda-1}$. *This is equivalent to* $\boxed{1 < \lambda < \frac{3}{2}}$.

ii. *The left-hand side of (4) is greater than the left-hand side of (3):*

$$-\frac{2}{3}\lambda + \frac{1}{3}\left(\frac{K_{20}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B} - \lambda\right) < -\frac{2}{3}\lambda + \frac{1}{3}\frac{K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B}.$$

This condition is equivalent to

$$\frac{K_{20}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B} - \lambda < \frac{K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B}$$
$$\text{or, } \lambda > \frac{K_{20}^B - K_{22}^B}{\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B}.$$

By definition, $K_{20}^B = D_{20}^B/\varphi_0^B$ *and* $K_{22}^B = D_{22}^B/\varphi_2^B$. *By symmetricity,*

$$K_{20}^B = D_{20}^B/\varphi_0^B = D_{10}^A/\varphi_0^A = 0.$$
$$K_{22}^B = D_{22}^B/\varphi_2^B = D_{11}^A/\varphi_1^A.$$

Inserting these, the condition becomes

$$\lambda > \frac{-D_{11}^A/\varphi_1^A}{D_{11}^A} = -\frac{1}{\varphi_0^A} = -\frac{1}{\frac{1}{2\lambda}} = -2\lambda$$

which is always satisfied.

And in this case, $D_{10}^A = 0$ *(since* $\varphi_0^A = 0$*),* $D_{11}^A = 1 - \frac{1}{\lambda}$*, and* $K_{11}^A = K_{22}^B = \frac{1-\frac{1}{\lambda}}{\frac{1}{2\lambda}} = 2(\lambda - 1)$.

(k) Complete the following table:

Table 4.

		Punishiment	φ_0^A	φ_1^A	D_{10}^A	D_{11}^A	K_{20}^B	K_{22}^B	Description
Case 1	$\lambda > \frac{3}{2}$	High							
Case 2a	$\frac{4}{3} < \lambda < \frac{3}{2}$	Medium High							
Case 2b	$1 < \lambda < \frac{4}{3}$	Medium Low							
Case 3	$\lambda < 1$	Low							

Answer: *In the following table, all the blanks are zero.*

		Punishiment	φ_0^A	φ_1^A	D_{10}^A	D_{11}^A	K_{20}^B	K_{22}^B	*Descrip*
Case 1	$\lambda > \frac{3}{2}$	*High*		$\frac{1}{3}$		$\frac{1}{3}$		1	*(See be*
Case 2a	$\frac{4}{3} < \lambda < \frac{3}{2}$	*Medium High*		$\frac{1}{2\lambda}$		$1-\frac{1}{\lambda}$		$2(\lambda - 1)$	*(See be*
Case 2b	$1 < \lambda < \frac{4}{3}$	*Medium Low*		$\frac{1}{2\lambda}$		$1-\frac{1}{\lambda}$		$2(\lambda - 1)$	*(See be*
			$\frac{1}{3\lambda}$		$1-\frac{1}{\lambda}$		$3(\lambda - 1)$		*(See be*
Case 3	$\lambda < 1$	*Low*							*(See be*

Case 1: Complete market with Assets #1 and #2.

Case 2a: Assets #1 and #2 traded, with partial delivery.

Case 2b: Multiple equilibria. There are two symmetric equilibria in which Agent A maximizes the expected utility given Agent B's strategy, and Agent B maximizes hers given Agent A's strategy.

Case 3: No market.

(l) In Case 2b in which $1 < \lambda < \frac{4}{3}$, there are two symmetric equilibria in which Agent A maximizes her expected utility given Agent B's choice, and Agent B maximizes hers given Agent A's choice. If the two agents can coordinate to pursue the symmetric equilibrim which provides a higher expected utility level to each, which equilibrium (out of the two) will be realized?

Answer: *First equilibrium with* $\varphi_0^A = 0$, $\varphi_1^A = \frac{1}{2\lambda}$,

$$\begin{aligned}
&\frac{2}{3}[\log(\frac{1}{\lambda}) - \lambda(\varphi_0^A + \varphi_1^A - 1 + \frac{1}{\lambda})] + \frac{1}{3}[\log(\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B) - \lambda\varphi_0^A] \\
&= \frac{2}{3}[\log(\frac{1}{\lambda}) - \lambda(\frac{1}{2\lambda} - 1 + \frac{1}{\lambda})] + \frac{1}{3}[\log(\frac{1}{2\lambda} \times 2(\lambda - 1))] \\
&= \frac{2}{3}[\log(\frac{1}{\lambda}) - \frac{3}{2} + \lambda] + \frac{1}{3}[\log(1 - \frac{1}{\lambda})] \\
&= \frac{2}{3}[\log(\frac{1}{\lambda}) + \lambda] + \frac{1}{3}[\log(1 - \frac{1}{\lambda})] - \frac{9}{4}
\end{aligned}$$

Second equilibrium with $\varphi_0^A = \frac{1}{3\lambda}$, $\varphi_1^A = 0$,

$$\begin{aligned}
&\frac{2}{3}[\log(\frac{1}{\lambda}) - \lambda(\varphi_0^A + \varphi_1^A - 1 + \frac{1}{\lambda})] + \frac{1}{3}[\log(\varphi_0^A \times K_{20}^B + \varphi_1^A \times K_{22}^B) - \lambda\varphi_0^A] \\
&= \frac{2}{3}[\log(\frac{1}{\lambda}) - \lambda(\frac{1}{3\lambda} - 1 + \frac{1}{\lambda})] + \frac{1}{3}[\log(\frac{1}{3\lambda} \times 3(\lambda - 1)) - \lambda\frac{1}{3\lambda}] \\
&= \frac{2}{3}[\log(\frac{1}{\lambda}) - \frac{4}{3} + \lambda] + \frac{1}{3}[\log(1 - \frac{1}{\lambda}) - \frac{1}{3}] \\
&= \frac{2}{3}[\log(\frac{1}{\lambda}) + \lambda] + \frac{1}{3}[\log(1 - \frac{1}{\lambda})] - 1
\end{aligned}$$

The second equilibrium dominates.

(m) Is the following statement TRUE or FALSE? Explain your reasoning. "The level of punishment (λ) on a default may affect the set of available assets in the asset market."

Answer: *Absolutely. For some value of* λ, *only Asset #0 is available. For some value, the market is complete and both Asset #1 and Asset #2 are available. For some value, there is no market at all.*

Issues with Preferences

References

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- [O] Bansal, R., and A. Yaron (2004), "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," Journal of Finance, 59(4), 1481-1509.

Puzzles (risk-free, equity-premium, excess-volatility) may be due to the assumptions on preferences.

1. CRRA

- CRRA (Constant Relative Risk Aversion) utility function:

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma},$$

  $\sigma > 0$ and $\sigma \neq 1$. As $\sigma \longrightarrow 1$, we have $u(C) \longrightarrow \log(C)$.

- What is σ?
- **Interpretation 1**: σ is a coefficient of **risk aversion**.
- The consumer have m units of physical goods.
- Consider a lottery: She receives an additional fraction x of m units: $E[x] = 0$, $var[x] = \sigma_x^2$.
- Expected utility: $E[u(m(1+x))]$
- Suppose for some constant γ, we have $u(\underbrace{m(1-\gamma)}_{\text{"certainty equivalence"}}) = E[u(m(1+x))]$.

- γ is called the **risk premium**.
- If more risk averse, γ is high. This is like if you are more risk averse, you are willing to pay more for auto insurance.
- Want: What is γ for CRRA?
- Answer: Taylor approximation: $f(y) \approx f(\alpha) + (y-\alpha)f'(\alpha) + \frac{1}{2}(y-\alpha)^2 f''(\alpha)$ (Please remember!)
- Apply this to the LHS, with $y = m - m\gamma$ and $\alpha = m$. Then,

$$\begin{aligned}
u(m - m\gamma) &\approx u(m) + ((m - m\gamma) - m)u'(m) + \frac{1}{2}((m - m\gamma) - m)^2 u''(m) \\
&= u(m) - m\gamma u'(m) + \frac{1}{2}(m\gamma)^2 u''(m) \\
&\text{(in case of CRRA)} \\
&= \frac{m^{1-\sigma}}{1-\sigma} - m\gamma m^{-\sigma} + \frac{1}{2}(m\gamma)^2(-\sigma)m^{-\sigma-1} \\
&= \frac{m^{1-\sigma}}{1-\sigma} - \gamma m^{1-\sigma} + \underbrace{\frac{1}{2}\gamma^2(-\sigma)m^{-\sigma+1}}_{\gamma^2 \text{ is relatively small.}} \\
&\approx \frac{m^{1-\sigma}}{1-\sigma} - \gamma m^{1-\sigma}
\end{aligned}$$

- Similarly for the RHS, with $y = m + mx$ and $\alpha = m$,

$$\begin{aligned}
E[u(m(1+x))] &\approx E[u(m) + ((m + mx) - m)u(m)' + \frac{1}{2}((m + mx) - m)^2 u''(m)] \\
&= E[u(m) + mxu'(m) + \frac{1}{2}(mx)^2 u''(m)] \\
&= u(m) + mu'(m)\underbrace{E[x]}_{=0} + \frac{1}{2}m^2 u''(m)\underbrace{E[x^2]}_{=\sigma_x^2} \\
&\text{(in case of CRRA)} \\
&= \frac{m^{1-\sigma}}{1-\sigma} + \frac{1}{2}m^2\left(-\sigma m^{-\sigma-1}\right)\sigma_x^2 \\
&= \frac{m^{1-\sigma}}{1-\sigma} - \frac{1}{2}\sigma\sigma_x^2 m^{1-\sigma}
\end{aligned}$$

- Hence,

$$\gamma m^{1-\sigma} \approx \frac{1}{2}\sigma\sigma_x^2 m^{1-\sigma}$$
$$\text{So } \frac{2\gamma}{\sigma_x^2} \approx \sigma$$

- So σ is proportional to γ, risk premium.
- **Interpretation 2**: σ is an inverse of the **intertemporal elasticity of substitution**.
- Utility: $u(C_1, C_2)$
- Elasticity of Substitution:

$$\frac{\%\text{ increase in } C_2/C_1}{\%\text{ increase in } p_1/p_2} = \frac{d\log(C_2/C_1)}{d\log(p_1/p_2)}$$

  (When good 1 becomes more expensive relative to good 2, how does good 2's consumption increase relative to good 1?)

- Measures the sensitivity of consumption given price changes.
- Utility: $u(C_0) + \beta u(C_1)$
- No uncertainty for simplicity.
- **Elasticity of Intertemporal Substitution**:

$$\frac{\%\text{ increase in } C_1/C_0}{\%\text{ increase in } p_0/p_1}$$

  where p_0 is the price of time-0 consumption and p_1 is the price of time-1 consumption in Arrow-Debreu.

- But

  time 0 —> time 1

  1 unit of physical good $(1 + r^f)$ units of physical goods

- So tomorrow's $(1 + r^f)$ units are equivalent to today's 1 unit.
- So $(1 + r^f)p_1 = p_0$.

- Now:

$$\frac{\%\text{ increase in } C_1/C_0}{\%\text{ increase in } (1+r^f)}$$

- Measures the sensitivity of intertemporal consumption given interest-rate changes.
- For CRRA,

$$\frac{1}{1+r^f} = \beta \left(\frac{C_1}{C_0}\right)^{-\sigma}.$$

- So EIS=

$$\frac{\%\text{ increase in } C_1/C_0}{\%\text{ increase in } \left(\frac{C_1}{C_0}\right)^{\sigma}} = \frac{d\log(C_1/C_0)}{d\log\left(\frac{C_1}{C_0}\right)^{\sigma}} = \frac{d\log(C_1/C_0)}{\sigma d\log\left(\frac{C_1}{C_0}\right)} = \frac{1}{\sigma}$$

- Conclusion: If σ increases,
- (i) Consumer is more risk averse.
- (ii) EIS decreases, which implies that consumer less sensitively react to a rise in interest rate.
- So what? Risk aversion and intertemporal substitution are separate concepts.
  - But our CRRA utility links them!
  - No problem in growth because there was no uncertainty anyway.
  - But now the uncertainty is our focus!

2. Resolutions

- Recursive utility (Epstein and Zin, 1991; Weil, 1991)
- Habit (Campbell and Cochrane, 1999)

EXERCISES

1. Are the following statements TRUE or FALSE? Provide your theoretical and/or empirical reasonings based on our class discussions. The credits will be solely based on your reasonings.

   (a) Suppose that the representative consumer has constant-relative-risk-aversion (CRRA) preferences. To be specific, suppose that she maximizes $E_0[\sum_{t=0}^{\infty} \beta^t u(C_t)]$, $0 < \beta < 1$, at time 0, where $u(c) = c^{1-\theta}/(1-\theta)$, $\theta > 0$ and $\theta \neq 1$. (And $u(c) = \log c$ if $\theta \longrightarrow 0$.) If θ increases, then the relative risk aversion increases, so the risk-free rate (i.e., the rate of return on risk-free asset) will decrease.

   Answer: *FALSE. It is true that if θ increases, the relative risk aversion increases. But an increase of θ also implies a decrease in the elasticity of intertemporal substitution. Then we have two offsetting effects:*

   (i) The relative risk aversion increases, so the consumer's demand on risk-free assets increases, which lowers the risk-free rate.

   (ii) The elasticity of intertemporal substitution decreases, so the consumer does not change her consumption plan sensitively according to the interest rate. The consumer needs a higher interest rate to keep a constant savings rate (which is 0 in this representative consumer's general equilibrium).

China's Stock Markets

1. Preliminaries

- 1984: 11 state-owned enterpreises became shareholding corporations.
- 1990: The first government-approved securities market, Shanghai Securities Exchange (SHSE), opened. There were mainly two types of stocks traded: **A shares**, bought and sold only by Chinese investors; and **B shares**, restricted to foreign investors.
- 1992: SHSE cancelled all the stock price limits.
- 1996: Facing the highly rise of the stock prices, the regulatory authorities reimposed a 10% daily price-change limit.
- 2002: Licensed foreign investors were allowed to buy and sell yuan-denominated A shares.
- 2006: Beijing allowed more than 1,300 listed companies to gradually sell their state-owned shares, effectively putting $270 billion in government-controlled assets in the public sector.
- **Limitations in research on China's stock market:** The market is immature. That is,
- (i) Data are for only 15 years.
- (ii) Even for those years, various regulations (e.g., restricting foreign investors from freely participating in A-share market, imposing price-change limits, etc.) have existed.
- (iii) The exchange rates between yuan and dollar have been controlled, which may have distorted investment decisions.
- Still, the second largest economy will become the world's largest in 2020s. The market is expected to grow, too. This implies that it will be important for practitioners and researchers to understand the market.

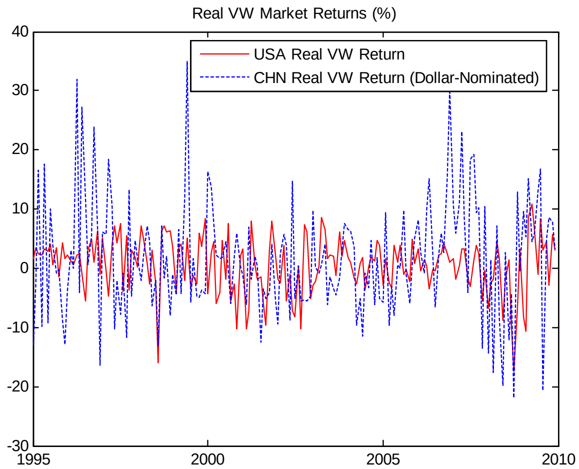
- **What Can Be Done? (Not complete remedies)**
- (i) Estimation techniques can be used for monthly data (about 180 monthly observations). Focus on medium-run trends based on descriptive statistics. ...
- (ii) Emphasize recent developments (e.g., reflecting increasing cross-sectional correlations between U.S. and China's stock returns) for simulations. ...
- (iii) Separately discuss the impacts of more flexible exchange rates. ...
- **Contents**

  1. U.S.-China Correlations
  2. Volatilities and Sharpe Ratios
  3. Cross-Section: Size Portfolios

- **Data**
  - The data on China's individual stock dividends and prices are from Wind (www.wind.com.cn). The data are from **December 1995 to July 2010**. China's aggregate value-weighted returns are obtained.
  - These **yuan-nominated returns** are transformed into the **dollar-nominated returns** using the exchange rate data from Board of Governors of the Federal Reserve System (http://research.stlouisfed.org/fred2/data/EXCHUS.txt).
  - The U.S. value-weighted returns are obtained from Kenneth French.
  - The U.S. inflation data are from BLS.

2. U.S.-China Correlations

- For the remaining of this note, all returns are dollar-nominated (based on exchange rates) and real (based on the U.S. CPI inflation rates), unless otherwise noted.
- **Figure**: USA and CHN Real VW Monthly Return



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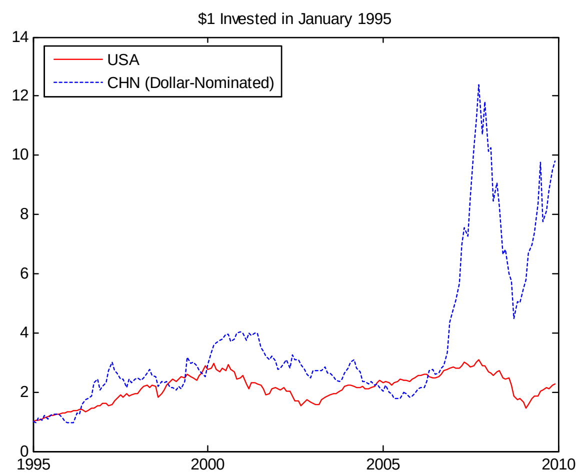
- **Figure**: $1 Invested in January 1995 (All returns reinvested)



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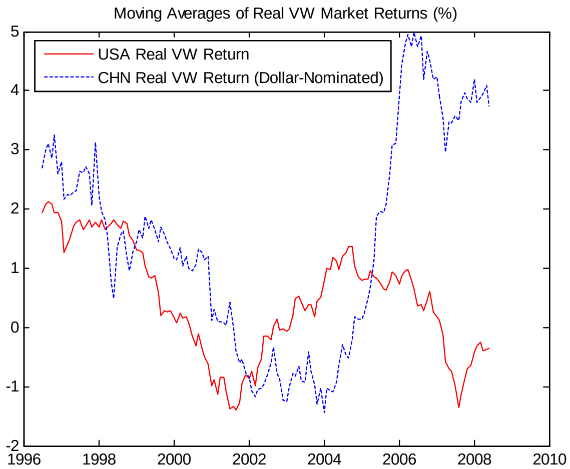
- **Figure**: 3Y (37M) moving averages of monthly returns:



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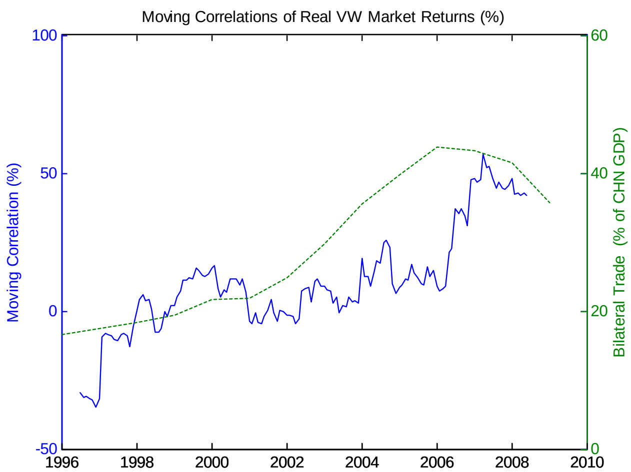
- **Lesson**:
  - (a) USA and CHN roughly move together (although there are some exceptional periods). We will confirm this soon.
  - 1996-2005: CHN followed the trend of USA, with a one-year lag.
  - 2005-06 is an exceptional period. (Maybe related to Beijing's decision to sell state-owned shares, but I am not sure.)
  - 2006-: CHN and USA move almost simulatenously.
  - (b) CHN experienced a huge rise in 2007-08, which quickly collapsed in one year or so. (Bubble?) Each of the following may or may not have been the contributors to this rise:
    * Government's selling state-owned shares
    * Injection of the liquidity, and other expansionary macro policies (e.g., facing natural disasters)
    * Response to a boom in real estate market
    * Better perspectives in Growth
    * Profits from international trade financially invested
    * Exchange rate reform: (i) Appriciation of RMB (ii) More flexible exchange rate scheme
    * Wide participation of the public in China in the stock market
    * International capital flows, especially from Japan
  - (c) CHN experienced another huge rise up to 2010. Not clear whether it will also collapse. This may be due to a **long-term reason** (e.g., **improved**

China's economic perspectives) or a **short-term reason** (e.g., temporary capital inflow into China).

- **Figure**: 3Y moving correlations, *along with bilateral trade as a share of China's GDP*



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- The solid line is "moving correlation" (left scale). The dotted line is "bilateral trade" (right scale).

- The correlation increases (except the last few years during the U.S. financial crisis). It is expected to increase further if USA and CHN economies are linked more closely.

- **Lesson**: USA and CHN have positive and increasing correlations.

  – One one hand, you want to avoid the risk. Stronger positive correlations imply that CHN is **less** attractive as your "insurance" against the U.S. risks.

- **Further Questions**

  1. The correlation increases. **Which component derives this correlation?**

     (a) $\frac{D_{t+1}}{P_t}$ or $\frac{P_{t+1}}{P_t}$?

     $$r_{t+1} = \frac{D_{t+1}}{P_t} + \frac{P_{t+1}}{P_t} - 1.$$

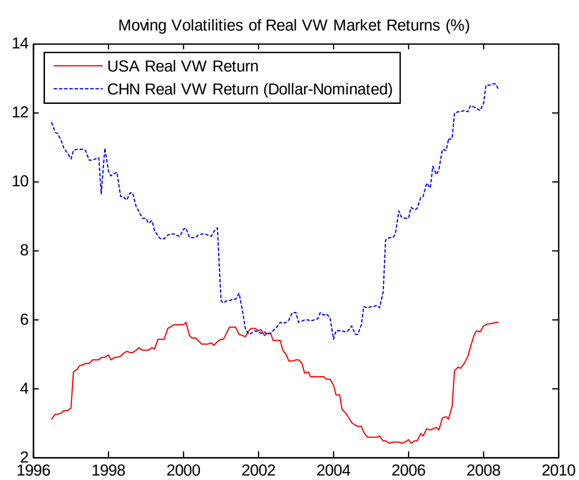
     (b) Or, study how dividend growths, P/D ratios, etc., co-move:

     $$\begin{aligned} r_{t+1} &= \frac{D_{t+1}}{D_t}\frac{D_t}{P_t} + \frac{P_{t+1}}{D_{t+1}}\frac{D_{t+1}}{D_t}\frac{D_t}{P_t} - 1 \\ &= \frac{D_{t+1}}{D_t}\frac{D_t}{P_t}(1 + \frac{P_{t+1}}{D_{t+1}}) - 1 \end{aligned}$$

2. Study how **capital inflows** to China are associated with China's returns.
3. **Exchange rate**. China's exchange rate has been controlled.
   (a) On one hand, this makes the U.S. investment in China less volatile, at least in the short run. (That is, the exchange-rate risk is gone.)
   (b) On the other hand, the official exchange rate may or may not reflect the market forces. If investors believe that China's official exchange rate will change shortly, the capital flows will be affected, which may also affect China's stock returns.
   (c) In the analysis of China's stock market, one may want to consider whether the current exchange rate is over-valued or under-valued? How? And when should I get my money out of China?
4. Apply "two Lucas trees" to international set-up. A shock in the U.S. dividend growth (or GDP growth) may affect the price of China's stocks. What is the mechanism?

3. Volatilities and Sharpe Ratios

- Volatilities also change over periods. Bad times tend to have higher volatilities in the U.S.
- **Figure**: 3Y moving volatilities of monthly returns:



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- Findings:
  - USA 3Y moving volatilities tend to be lower when USA 3Y moving averages are higher.

    * 1996-2002: USA average decreased. USA volatility increased.
    * 2002-2006: USA average increased. USA volatility decreases.
    * 2007-: USA average decreased. USA volatilities increases.
  - Near 1996, CHN was young, small market, which implies a decision of a small group of investors may heavily affect the market prices. So it faced higher volatilities.
  - 2006-: CHN and USA co-move for volatilities?

- **Lesson**: Volatilities seem to move together after 2005, and CHN volatilities are higher than USA. CHN is "good" since average returns are higher, but is "bad" since volatilities are also higher.

- How do we compare the "good" and "bad"?

- **Descriptive Statistics** on monthly returns:

	USA	USA	CHN
	(1950:1-2009:12)	(1995:1-2009:12)	(1995:1-2009:12)
Average	0.7%	0.6%	1.6%
Volatility	4.3%	4.8%	9.7%
Sharpe Ratio	0.13	0.11	0.15

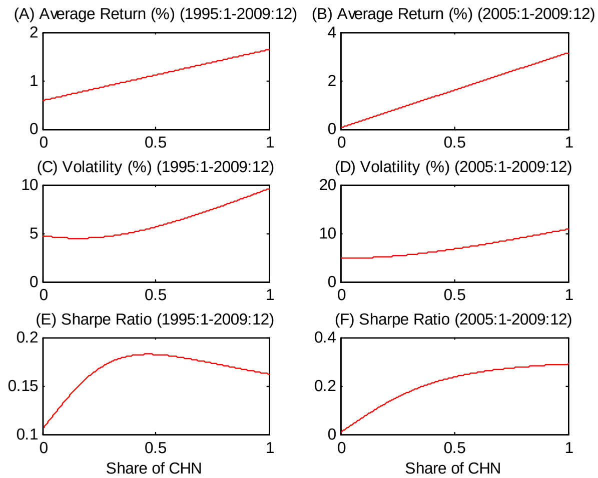
- **Sharpe ratio**
  - $=\frac{\text{TS Average Excess Return Relative to Risk-free Return}}{\text{Volatility of Excess Return Relative to Risk-free Return}}=\frac{\text{TS Average of }(r_t^j-r_t^f)}{\text{Volatility of }(r_t^j-r_t^f)}$.
  - r_t^f is proxed by the one-month return on the U.S. treasury bills.
  - Simple measure about how much the asset pays compared to the risk.
  - To understand the Sharpe ratio better, consider an investor forming a portfolio between two assets: risk-free asset and asset j.
  - Assumption: He prefers higher mean return (expected return). He prefers lower volaility of return. (But **consumption covariance**, etc., is disregarded. That is, this analysis needs further work reflecting the advances in modern asset pricing theory.)
  - Mean |
  - | * (Asset j)
  - |
  - |

- * Risk-free (Next period's return is certain, so the volatility is zero)
- |
- +——————————————- Volatility (Standard Deviation)
- r^j_{t+1} has a mean μ^j and volatility (standard deviation) σ^j.
- Suppose that in his portfolio, he weighs $1-\omega$ to risk-free asset and ω to asset j. This portfolio's return is $(1-\omega)r^f + \omega r^j_{t+1}$.
- Mean: $E_t[(1-\omega)r^f + \omega r^j_{t+1}] = (1-\omega)r^f + \omega\mu^j$. So it is the average of two expected returns weighted by portfolio weights.
- Volatility:

$$\begin{aligned}
\sqrt{var_t[(1-\omega)r^f + \omega r^j_{t+1}]} &= \sqrt{\underbrace{var_t[(1-\omega)r^f]}_{=0} + var_t[\omega r^j_{t+1}]} \\
&= \sqrt{\omega^2 var_t[r^j_{t+1}]} \\
&= \omega\sqrt{var_t[r^j_{t+1}]} \\
&= \omega\sigma^j
\end{aligned}$$

- So mean and volatility of this portfolio is ω times mean and volatility of asset j.
- Imagine a straight line between the two points in the figure. (It can go further to Northeast from Asset j's point ($\omega = 1$). If $\omega > 1$, then this investor "short-sells" the risk-free asset.) This is a **feasible set** (or a budget constraint). The investor will choose one point based on his indifference curve.
- Notice that the slope of this constraint is the **Sharpe ratio**.
- Now suppose that there are many assets. The asset with the highest Sharpe ratio will be chosen because that gives the best feasible set.

- **Figure**: Historical Average, Volatility and Sharpe Ratio for USA-CHN portfolio (x-axis: Share of CHN)



- Interpretation (1995-2009)
  - Average return for USA-CHN portfolio is a linear combination of USA average return and CHN average return.
  - The volatility is minimized at s_t around 0.2. This is the gain from **diversification**.
  - The Sharpe ratio is maximized at s_t around 0.45.

- **Lesson**: Historical Sharpe ratios suggest that it may be helpful to have both USA and CHN if you are a mean-variance investor (i.e., prefer a higher Sharpe ratio).

- The historical Sharpe ratio is useful, but the correlation between USA and CHN is increasing in data.

- Let's apply higher correlation.

- **Attempt 1 (simplest possible model)**: Generate r_t^{US}˜iid $N(0.6\%, 4.8\%)$. (The numbers are historical observations for 1995:1-2009:12.)

- Then $r_t^{CH} = \mu + \alpha r_t^{US} + \varepsilon_t$, where ε_t˜iid $N(0, \sigma)$, independent of r_t^{US}. Match r_t^{CH} with mean 1.6% and volatility 9.7% (historical observations for 1995:1-2009:12), and $\rho(r_t^{US}, r_t^{CH}) = 0.4$ **(based on recent, higher observations on the correlation)**.
  - First,

$$\begin{aligned} cov[r_t^{US}, r_t^{CH}] &= cov[r_t^{US}, \alpha r_t^{US} + \varepsilon_t] = cov[r_t^{US}, \alpha r_t^{US}] + cov[r_t^{US}, \varepsilon_t] \\ &= \alpha var[r_t^{US}] + 0 = \alpha 0.048^2 \end{aligned}$$

Hence,

$$\rho(r_t^{US}, r_t^{CH}) = \frac{cov[r_t^{US}, r_t^{CH}]}{sd[r_t^{US}]sd[r_t^{CH}]} = \frac{\alpha 0.048^2}{0.048 \times 0.097} = \frac{0.048}{0.097}\alpha = 0.4$$

This gives

$$\alpha = 0.4\frac{0.097}{0.048} = 0.808$$

– Second,

$$E[r_t^{CH}] = \mu + \alpha E[r_t^{US}] = \mu + 0.006\alpha = 0.016$$

Hence,

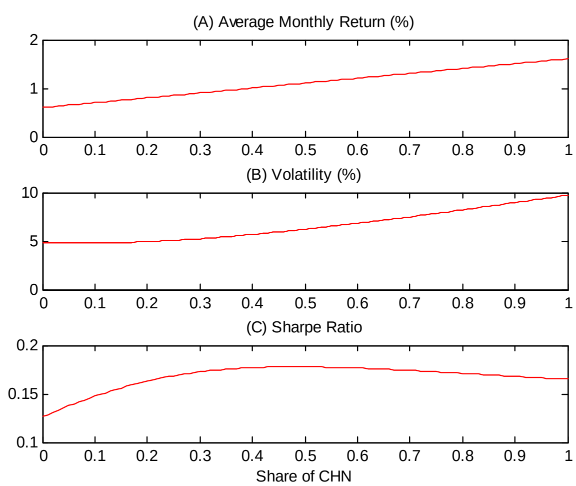
$$\mu = 0.016 - 0.006 \times 0.808 = 0.011$$

– Third,

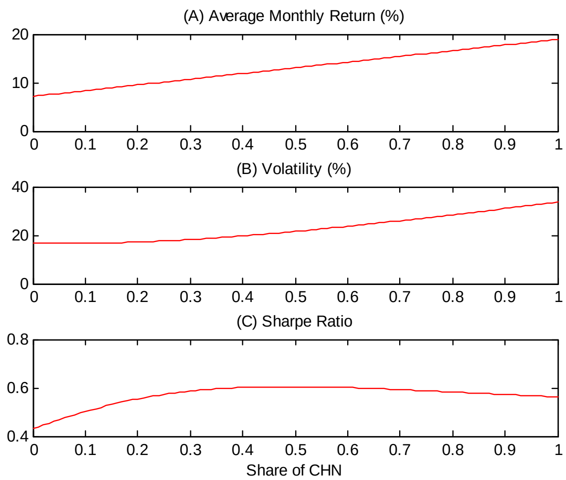
$$var[r_t^{CH}] = \alpha^2 var[r_t^{US}] + var[\varepsilon_t] = \alpha^2 0.048^2 + \sigma^2 = 0.097^2.$$

Hence,

$$\sigma = \sqrt{0.097^2 - \alpha^2 0.048^2} = \sqrt{0.097^2 - 0.808^2 0.048^2} = 0.089$$

- Monte Carlo: Based on 100,000 simulations. (Need to prove analytically!)
- **Figure**: Simulated Sharpe Ratio, for monthly returns
- 

- **Figure**: Simulated Sharpe Ratio, for **annual** returns (An annual return is the sum of 12 monthly returns)

- 

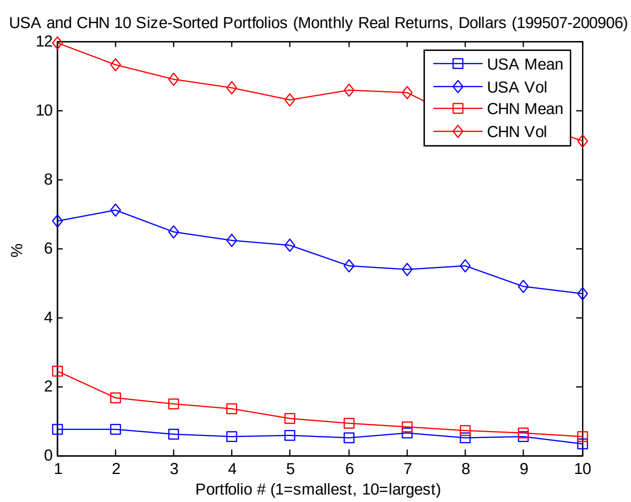
- Implication: s_t =about 0.5 maximizes the Sharpe ratio, regardless of the holding period (same for 10 years). (The holding period does not matter because returns are i.i.d. over time, which is somewhat unrealistic)

- **Further Questions**

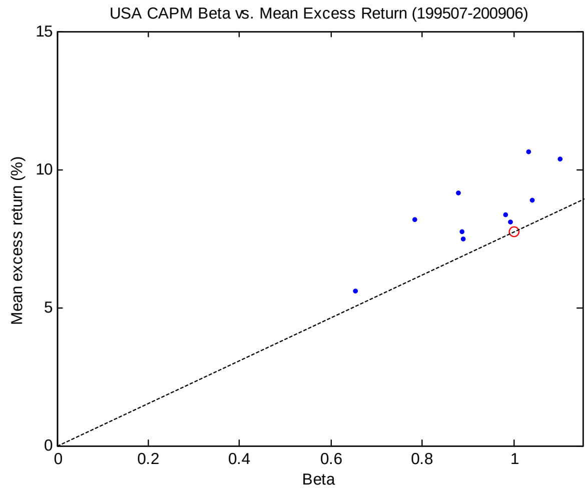
  1. Attempts 2, 3, ...: Make the model more realisitic. (That is, introducing autocorrelation, etc.)
  2. Further implications on portfolio decisions? (We might use Brandt's (1999) optimal potfolio decision at period t in which T is the end of the world.)

4. Cross-Section: Size Portfolios

- **Figure**: 10 Size-Sorted Portfolios

- 

- Finding: CHN also has the size effect.
- **Figure**: Application of the CAPM to USA



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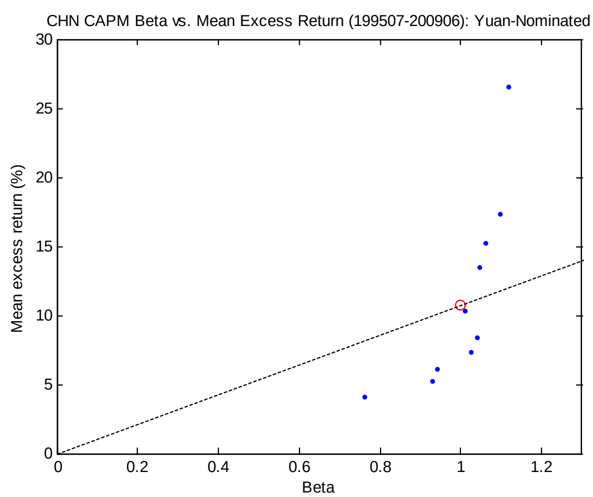
- **Figure**: Application of the CAPM to CHN: Based on **yuan-nominated** returns, using China's "deposite interest rate" as risk-free.



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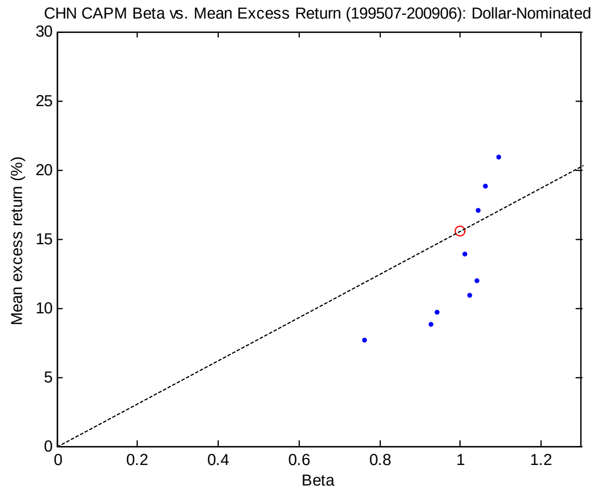
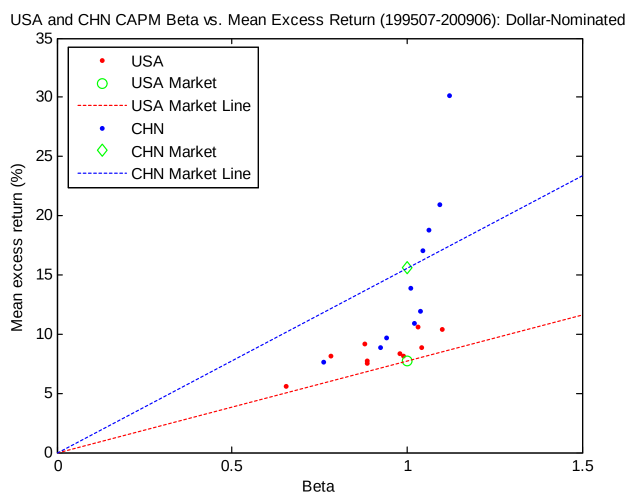
- **Figure**: Application of the CAPM to CHN: Based on **dollar-nominated** returns, using CHN's market return but USA risk-free rate.



- **Figure**: Two dollar-nominated figures at the same time:



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- Lesson: CHN small portfolios perform great, IF (i) the CAPM is the right model, and (ii) the trends in past observations will continue in the future.

- More work: All CHN portfolios need to be run with USA market returns to make them directly comparable to USA portfolios.

- **Further Questions**

  1. BTM-sorted portfolios? FF25? Portfolios sorted in other ways? Industry-level portfolios? ...

  2. Does CAPM work in China? Does FF3F model work in China?

(a) Fama-MacBeth procedure, or some other tests.
(b) Are there some other explanatory variables explaining the cross-section of returns, which are especially important for China, but not for the U.S.?